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More Sets, Graphs and Numbers

A Salute to Vera Sós and András Hajnal

Springer
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Preface

The present volume is slightly connected to the conference organized in Budapest, January 2001 to the honour of Vera Sós and András Hajnal on the occasion of their 70th birthdays. Namely, we mainly asked the invited speakers of that conference to write survey papers on their favorite subjects. Therefore the volume contains strong and well-written surveys in the areas of the celebrated colleagues: mostly in combinatorics, graph theory, less in number theory and set theory. The authors gave the up-to-date state of the art in their subjects, put the recent results into integral framework. Examples are listed below. The other papers contain original research results.

Matthias Beck, Xueqin Wang, and Thomas Zaslavsky find a nice, so-called unifying generalization of different versions of Sperner's theorem. They found a uniform handling of several different generalizations.

Béla Bollobás and Alexander Scott summarize different results on discrepancies of graphs and hypergraphs.

Éva Czabarka, Ondrej Sýkora, László A. Székely and Imrich Vrto survey some bounds on biplanar crossing numbers of graphs which is the sum of the crossing numbers over all partitions of a graph into two planar graphs.

András Frank studies the different notions of edge-connectivity of graphs, digraps and hypergraphs and uses properties of submodular functions to get different theorems on them. He gives an extensive survey of the results concerning orientations and connectivity augmentations in a general setting.

Kálmán Győry surveys when we can get (almost) complete powers as the product of consecutive terms of an arithmetic progression or binomial coefficients. The results are mostly negative as it turns out from the nice overview of classical papers of Erdős and Selfridge as well as the recent ones of the surveyer and others.

István Juhász and Andrzej Szymanski present a purely topological generalization of Fodor's theorem called "the pressing down lemma". By means of it, the authors prove a partial generalization of this framework of Solovay's celebrated stationary set decomposition theorem.

In his extensive survey paper, Alexandr Kostochka summarizes the results on the minimum number of edges in color-critical graphs and hypergraphs.

Michael Krivelevich and Benny Sudakov give an extensive survey on pseudo random graphs with emphasis on the results obtained by means of the investigation of the eigenvalues of the adjacency matrix.

Jaroslav Nešetřil deals with questions and results concerning order-theoretic properties of the homomorphism order of graphs, but the author surveys upper bounds, suprema and maximal elements of the homomorphism order lattice in other interesting finite structures too. The author also studies minor closed classes of graphs, shows how the order setting captures Hadwiger conjecture and suggests some new problems too.

András Recski and Dávid Szeszlér investigate VLSI routing algorithms, especially the influence of Gallai's Algorithm on them. They show the first forty years of the influence on VLSI design of the classic result on the perfectness of interval graphs.

András Sárközy's paper describes advance in a specific question, the possible behaviour of representation functions. We take a set A of positive integers, and consider $r_k(n)$, the number of representations of n as a sum of k elements of A, or variants where the order is neglected or where an element can be used only once. Typical questions are whether such a function can be monotonic, or can be very near to a given regular function. The author presents plenty of results and unsolved problems.

Andrew Thomason presents results and methods concerning the minimum number of edges guaranteeing a given graph minor. It turns out that the extremal graphs are pseudo-random. The survey describes what is known about the extremal function and discusses some related matters.

Robert Tijdeman's survey covers a broad area, with main emphasis on tilings and balanced words. We learn how words with small complexity (that is, with a small number of different subwords of length n for every n) are connected with balanced words, where the number of occurrences of any fixed letter in subwords of given length is almost constant, and with sequences given by the integer part of a linear function.

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The editors

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A Unifying Generalization of Sperner's Theorem

M. BECK, X. WANG and T. ZASLAVSKY*

Dedicated to the memories of Pál Erdős and Lev Meshalkin

Sperner's bound on the size of an antichain in the lattice $\mathcal{P}(S)$ of subsets of a finite set S has been generalized in three different directions: by Erdős to subsets of $\mathcal{P}(S)$ in which chains contain at most r elements; by Meshalkin to certain classes of compositions of S; by Griggs, Stahl, and Trotter through replacing the antichains by certain sets of pairs of disjoint elements of $\mathcal{P}(S)$. We unify these three bounds with a common generalization. We similarly unify their accompanying LYM inequalities. Our bounds do not in general appear to be the best possible.

1. Sperner-type theorems

Let S be a finite set with n elements. In the lattice $\mathcal{P}(S)$ of all subsets of S one tries to estimate the size of a subset with certain characteristics. The most famous such estimate concerns **antichains**, that is, subsets of $\mathcal{P}(S)$ in which any two elements are incomparable.

Theorem 1.1 (Sperner [11]). *Suppose $A_1, \ldots, A_m \subseteq S$ such that $A_k \nsubseteq A_j$ for $k \neq j$. Then $m \leq \binom{n}{\lfloor n/2 \rfloor}$. Furthermore, this bound can be attained for any n.*

We attain the bound by taking all $\lfloor \frac{n}{2} \rfloor$-element subsets of S, or all $\lceil \frac{n}{2} \rceil$-element subsets, but in no other way. There are many ways to prove

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Sperner's bound and the near-uniqueness of the maximal example; several of them will be found in the opening chapters of Anderson's lovely introductory book [1]. The most famous approach is perhaps that of the "LYM inequality"; see Theorem 2.1 below.

Sperner's theorem has been generalized in many different directions. Here are three: Erdős extended Sperner's inequality to subsets of $\mathcal{P}(S)$ in which chains contain at most r elements. Meshalkin proved a Sperner-like inequality for families of compositions of S into a fixed number of parts, in which the sets in each part constitute an antichain. Finally, Griggs, Stahl, and Trotter extended Sperner's theorem by replacing the antichains by sets of pairs of disjoint elements of $\mathcal{P}(S)$ satisfying an intersection condition. In this paper we unify the Erdős, Meshalkin, and Griggs–Stahl–Trotter inequalities in a single generalization. However, except in special cases (among which are generalizations of the known bounds), our bounds are not the best possible.

For a precise statement of Erdős's generalization, call a subset of $\mathcal{P}(S)$ **r-chain-free** if its chains (i.e., linearly ordered subsets) contain no more than r elements; that is, no chain has length r.[1] In particular, an antichain is 1-chain-free. The generalization of Theorem 1.1 to r-chain-free families is

Theorem 1.2 (Erdős [4]). *Suppose $\{A_1, \dots, A_m\} \subseteq \mathcal{P}(S)$ contains no chains with $r+1$ elements. Then m is bounded by the sum of the r largest binomial coefficients $\binom{n}{k}$, $0 \le k \le n$. The bound is attainable for every n and r.*

Sperner's theorem is the case $r = 1$. To attain the bound take all subsets of sizes $\lfloor \frac{n-r+1}{2} \rfloor \le k \le \lfloor \frac{n+r-1}{2} \rfloor$ or all of sizes $\lceil \frac{n-r+1}{2} \rceil \le k \le \lceil \frac{n+r-1}{2} \rceil$; these are the only ways.

Going in a different direction, Sperner's inequality can be generalized to certain ordered weak partitions of S. We define a **weak partial composition of S into p parts** as an ordered p-tuple $(A_1, \dots, A_p)$ of sets A_k, possibly void (hence the word "weak"), such that $A_1, \dots, A_p$ are pairwise disjoint and $A_1 \cup \dots \cup A_p \subseteq S$. If $A_1 \cup \dots \cup A_p = S$, we have a **weak composition** of S. A Sperner-like inequality suitable for this setting was proposed by Sevast'yanov and proved by Meshalkin (see [9]). By a **p-multinomial coefficient for n** we mean a multinomial coefficient $\binom{n}{a_1, \dots, a_p}$, where $a_i \ge 0$ and $a_1 + \dots + a_p = n$. Let $[p] := \{1, 2, \dots, p\}$.

[1] The term "r-family" or "k-family", depending on the name of the forbidden length, has been used in the past, but we think it is time for a distinctive name.

Theorem 1.3 (Meshalkin). *Let $p \geq 2$. Suppose $(A_{j1}, \dots, A_{jp})$ for $j = 1, \dots, m$ are different weak compositions of S into p parts such that, for each $k \in [p]$, the set $\{A_{jk} : 1 \leq j \leq m\}$ (ignoring repetition) forms an antichain. Then m is bounded by the largest p-multinomial coefficient for n. Furthermore, the bound is attainable for every n and p.*

This largest multinomial coefficient can be written explicitly as

$$\frac{n!}{\left(\left(\lfloor \frac{n}{p} \rfloor + 1\right)!\right)^{\rho} \left(\lfloor \frac{n}{p} \rfloor !\right)^{p-\rho}},$$

where $\rho = n - p\lfloor \frac{n}{p} \rfloor$. We attain the bound by choosing any set $K \subseteq [p]$ of size ρ and taking all weak compositions $(A_{j1}, \dots, A_{jp})$ in which $|A_{jk}| = \lfloor \frac{n}{p} \rfloor$ if $k \in K$ and $|A_{jk}| = \lceil \frac{n}{p} \rceil$ if $k \notin K$. Hochberg and Hirsch [6] showed that no other family of weak compositions of S has maximum size. Meshalkin's theorem and the completion by Hochberg and Hirsch are curiously neglected: we have not seen them mentioned in any book except [7].

To see why Meshalkin's inequality generalizes Sperner's Theorem, suppose $A_1, \dots, A_m \subseteq S$ form an antichain. Then $S - A_1, \dots, S - A_m$ also form an antichain. Hence the m weak compositions $(A_j, S - A_j)$ of S into two parts satisfy Meshalkin's conditions and Sperner's inequality follows.

Yet another generalization of Sperner's Theorem is

Theorem 1.4′ (Griggs–Stahl–Trotter [5]). *Suppose $\{A_{j0}, \dots, A_{jq}\}$ for $j = 1, \dots, m$ are chains of size $q + 1$ in $\mathcal{P}(S)$ such that $A_{ji} \not\subseteq A_{kl}$ for all i and l and all $j \neq k$. Then $m \leq \binom{n-q}{\lfloor (n-q)/2 \rfloor}$. Furthermore, this bound can be attained for all n and q.*

An equivalent, simplified form of this result (in which $A_j = A_{j0}$, $B_j = S - A_{jq}$, and n replaces $n - q$) is

Theorem 1.4. *Let $n > 0$. Suppose (A_j, B_j) are m pairs of sets such that $A_j \cap B_j = \varnothing$ for all j, $A_j \cap B_k \neq \varnothing$ for all $j \neq k$, and all $|A_j| + |B_j| \leq n$. Then $m \leq \binom{n}{\lfloor n/2 \rfloor}$ and this bound can be attained for every n.*

Sperner's inequality follows as the special case in which $A_1, \dots, A_m \subseteq S$ form an antichain and $B_j = S - A_j$. To attain the bound in Theorem 1.4′ take $\{A_{j0}\}$ to consist of all subsets of $[n - q]$ of size $\lfloor \frac{n-q}{2} \rfloor$, or all of size $\lceil \frac{n-q}{2} \rceil$. Then let $A_{jk} = A_{j0} \cup \{n - q + 1, \dots, n - q + k\}$. In Theorem 1.4, take $A_j = A_{j0}$ and $B_j = [n] - A_j$.

Theorems 1.2, 1.3, and 1.4 are *incomparable* generalizations of Sperner's Theorem. We wish to combine (and hence further generalize) these generalizations. To state our main result, we define a **weak set composition** as a weak composition of any set S. Our generalization of Sperner's inequality is:

Theorem 1.5. *Fix integers $p \geq 2$ and $r \geq 1$. Suppose $(A_{j1}, \ldots, A_{jp})$ for $j = 1, \ldots, m$ are different weak set compositions into p parts with the condition that, for all $k \in [p]$ and all $I \subseteq [m]$ with $|I| = r + 1$, there exist distinct $i, j \in I$ such that either $A_{ik} = A_{jk}$ or*

$$A_{ik} \cap \bigcup_{l \neq k} A_{jl} \neq \varnothing \neq A_{jk} \cap \bigcup_{l \neq k} A_{il}, \tag{1}$$

and let $n := \max_{1 \leq j \leq m} \left(|A_{j1}| + \cdots + |A_{jp}| \right)$. Then m is bounded by the sum of the r^p largest p-multinomial coefficients for integers less than or equal to n.

Think of the p-multinomial coefficients as a sequence arranged in weakly descending order. Then if r^p is larger than $\binom{r+p}{p}$, the number of p-multinomial coefficients, we regard the sequence of coefficients as extended by 0's.

The reader may find the statement of this theorem somewhat difficult. We would first like to show that it does generalize Theorems 1.2, 1.3, and 1.4 simultaneously. The last follows easily as the case $r = 1$, $p = 2$. Theorem 1.3 can be deduced by choosing $r = 1$ and restricting the weak compositions to be compositions of a fixed set S with n elements. Finally, Theorem 1.2 follows by choosing $p = 2$ and the weak compositions to be compositions of a fixed n-set into 2 parts. What we find most interesting, however, is that specializing Theorem 1.5 yields three corollaries that generalize two at a time of Theorems 1.2, 1.3, and 1.4 yet are easy to state and understand. Section 4 collects these corollaries.

We came to Theorem 1.5 through seeking a common generalization of Erdős's and Meshalkin's theorems (see Corollary 4.1); our original motivation was, in part, surprise at the lack of general awareness of Meshalkin's result. When we learned of the Griggs–Stahl–Trotter theorem, we could not be satisfied until we succeeded in extending our result to include it as well. (Fortunately for us, we did not encounter a fourth kind of Sperner generalization.)

The condition of the theorem implies that each set $\mathcal{A}_k = \{ A_{jk} : j \in [m] \}$ (ignoring repetition) is r-chain-free. We suspect that the converse is not true in general. (It is true if all the weak set compositions are weak compositions of the same set of order n, as in Corollary 4.1.)

All the theorems we have stated have each a slightly stronger companion, an *LYM inequality*. In Section 2, we state these inequalities and show how Theorems 1.1–1.5 can be deduced from them. The proofs of Theorem 1.5 and the corresponding LYM inequality are in Section 3. After the corollaries of Section 4, in Section 5 we show that some, at least, of our upper bounds cannot be attained.

2. LYM INEQUALITIES

In attempting to estimate the order of the free distributive lattice with n generators, Yamamoto came up with the following result, which was rediscovered by Meshalkin in the course of proving his Sperner generalization (Theorem 1.3) and still later by Lubell with a classic short proof. In the meantime Bollobás had independently proved even a generalization (Theorem 2.4 below). The result is the famous LYM inequality, that has given its name to a whole class of similar relations.

Theorem 2.1 (Yamamoto [12, §6], Meshalkin [9, Lemma], Lubell [8]). *Suppose* $A_1, \ldots, A_m \subseteq S$ *such that* $A_k \not\subseteq A_j$ *for* $k \neq j$. *Then*

$$\sum_{k=1}^{m} \frac{1}{\binom{n}{|A_k|}} \leq 1.$$

Sperner's inequality follows immediately by noting that $\max_k \binom{n}{k} = \binom{n}{\lfloor n/2 \rfloor}$.

An LYM inequality corresponding to Theorem 1.2 appeared to our knowledge first in [10]:

Theorem 2.2 (Rota–Harper). *Suppose* $\{A_1, \ldots, A_m\} \subseteq \mathcal{P}(S)$ *contains no chains with* $r+1$ *elements. Then*

$$\sum_{k=1}^{m} \frac{1}{\binom{n}{|A_k|}} \leq r.$$

Deducing Erdős's Theorem 1.2 from this inequality is not as straightforward as the connection between Theorems 2.1 and 1.1. It can be done through Lemma 3.1, which we also need in order to deduce Theorem 1.5.

The LYM companion of Theorem 1.3 first appeared in [6]; again, Meshalkin's Theorem 1.3 follows immediately.

Theorem 2.3 (Hochberg–Hirsch). *Suppose* $(A_{j1}, \dots, A_{jp})$ *for* $j = 1, \dots, m$ *are different weak compositions of* S *into* p *parts such that for each* $k \in [p]$ *the set* $\{A_{jk} : 1 \le j \le m\}$ *(ignoring repetitions) forms an antichain. Then*

$$\sum_{j=1}^{m} \frac{1}{\binom{n}{|A_{j1}|,\dots,|A_{jp}|}} \le 1.$$

The LYM inequality corresponding to Theorem 1.4 is due to Bollobás.

Theorem 2.4 (Bollobás [3]). *Suppose* (A_j, B_j) *are* m *pairs of sets such that* $A_j \cap B_j = \varnothing$ *for all* j *and* $A_j \cap B_k \ne \varnothing$ *for all* $j \ne k$. *Then*

$$\sum_{j=1}^{m} \frac{1}{\binom{|A_j|+|B_j|}{|A_j|}} \le 1.$$

Once more, the corresponding upper bound, the Griggs–Stahl–Trotter Theorem 1.4, is an immediate consequence.

Naturally, there is an LYM inequality accompanying our main Theorem 1.5. Like its siblings, it constitutes a refinement.

Theorem 2.5. *Let* $p \ge 2$ *and* $r \ge 1$. *Suppose* $(A_{j1}, \dots, A_{jp})$ *for* $j = 1, \dots, m$ *are different weak compositions (of any sets) into* p *parts satisfying the same condition as in Theorem 1.5. Then*

$$\sum_{j=1}^{m} \frac{1}{\binom{|A_{j1}|+\cdots+|A_{jp}|}{|A_{j1}|,\dots,|A_{jp}|}} \le r^p.$$

Example 2.1. The complicated hypothesis of Theorem 2.5 cannot be replaced by the assumption that each $\mathcal{A}_k$ is r-chain-free, because then there is no LYM bound independent of n. Let $n \gg p \ge 2$, $S = [n]$, and $\mathcal{A} = \big\{ \big(A, \{n\}, \{n-1\}, \dots, \{n-p+2\} \big) : A \in \mathcal{A}_1 \big\}$ where $\mathcal{A}_1$ is a largest r-chain-free family in $[n-p+1]$, specifically,

$$\mathcal{A}_1 = \bigcup_{j \in I} \mathcal{P}_j\big([n-p+1]\big)$$

where

$$I = \left\{ \left\lceil \frac{n-p+1-r}{2} \right\rceil, \left\lceil \frac{n-p+1-r}{2} \right\rceil + 1, \left\lceil \frac{n-p+1-r}{2} \right\rceil + r - 1 \right\}.$$

The LYM sum is

$$\begin{aligned}
\sum_{A \in \mathcal{A}_1} \frac{1}{\binom{|A|+p-1}{|A|,1,\ldots,1}} &= \sum_{A \in \mathcal{A}_1} \frac{|A|!}{\big(|A|+p-1\big)!} \\
&= \sum_{j \in I} \binom{n-p+1}{j} \frac{j!}{(j+p-1)!} \\
&= \sum_{j \in I} \frac{(n-p+1)\cdots(n-p-j+2)}{(p-1+j)!} \\
&\to \infty \quad \text{as} \quad n \to \infty.
\end{aligned}$$

There is no possible upper bound in terms of n.

3. Proof of the main theorems

Proof of Theorem 2.5. Let S be a finite set containing all A_{jk} for $j = 1, \ldots, m$ and $k = 1, \ldots, p$, and let $n = |S|$. We count maximal chains in $\mathcal{P}(S)$. Let us say a maximal chain **separates** the weak composition $(A_1, \ldots, A_p)$ if there exist elements $\varnothing = X_0 \subseteq X_{l_1} \subseteq \cdots \subseteq X_{l_p} = S$ of the maximal chain such that $A_k \subseteq X_{l_k} - X_{l_{k-1}}$ for each k. There are

$$\binom{n}{|A_1| + \cdots + |A_p|} |A_1|! \cdots |A_p|! \big(n - |A_1| - \cdots - |A_p|\big)! \tag{2}$$

maximal chains separating $(A_1, \ldots, A_p)$. (To prove this, replace maximal chains $\varnothing \subset \{x_1\} \subset \{x_1, x_2\} \subset \cdots \subset S$ by permutations $(x_1, x_2, \ldots, x_n)$ of S. Choose $|A_1| + \cdots + |A_p|$ places for $A_1 \cup \cdots \cup A_p$; then arrange A_1 in any order in the first $|A_1|$ of these places, A_2 in the next $|A_2|$, etc. Finally, arrange $S - (A_1 \cup \cdots \cup A_p)$ in the remaining places. This constructs all maximal chains that separate $(A_1, \ldots, A_p)$.)

We claim that every maximal chain separates at most r^p weak partial compositions of $|S|$. To prove this, assume that there is a maximal chain that separates N weak partial compositions $(A_{j1}, \dots, A_{jp})$. Consider all first components A_{j1} and suppose $r+1$ of them are different, say $A_{11}, A_{21}, \dots, A_{r+1,1}$. By the hypotheses of the theorem, there are $i, i' \in [r+1]$ such that A_{i1} meets some $A_{i'l'}$ where $l' > 1$ and $A_{i'1}$ meets some A_{il} where $l > 1$. By separation, there are q_1 and q'_1 such that $A_{i1} \subseteq X_{q_1} - X_0$ and $A_{i'1} \subseteq X_{q'_1} - X_0$, and there are $q_{l-1}, q_l, q'_{l'-1}, q'_{l'}$ such that $q_1 \leq q_{l-1} \leq q_l$, $q'_1 \leq q'_{l'-1} \leq q'_{l'}$, and

$$A_{il} \subseteq X_{q_l} - X_{q_{l-1}} \qquad \text{and} \qquad A_{i'l'} \subseteq X_{q'_{l'}} - X_{q'_{l'-1}}.$$

Since A_{i1} meets $A_{i'l'}$, there is an element $a_{i1} \in X_{q'_{l'}} - X_{q'_{l'-1}}$; it follows that $q'_{l'-1} < q_1$. Similarly, $q_{l-1} < q'_1$. But this is a contradiction. It follows that, amongst the N sets A_{j1}, there are at most r different sets. Hence (by the pigeonhole principle) there are $\lceil N/r \rceil$ among the N weak partial compositions that have the same first set A_{j1}.

Looking now at these $\lceil N/r \rceil$ weak partial compositions, we can repeat the argument to conclude that there are $\lceil \lceil N/r \rceil / r \rceil \geq \lceil N/r^2 \rceil$ weak partial compositions for which both the A_{j1}'s and the A_{j2}'s are identical. Repeating this process $p-1$ times yields $\lceil N/r^{p-1} \rceil$ weak partial compositions into p parts whose first $p-1$ parts are identical. But now the hypotheses imply that the last parts of all these weak partial compositions are at most r different sets; in other words, there are at most r distinct weak partial compositions. Hence $\lceil N/r^{p-1} \rceil \leq r$, whence $N \leq r^p$. (If we know that all the compositions are weak—but not partial—compositions of S, then the last parts of all these $\lceil N/r^{p-1} \rceil$ weak compositions are identical. Thus $N \leq r^{p-1}$.)

Since at most r^p weak partial compositions of S are separated by each of the $n!$ maximal chains, from (2) we deduce that

$$r^p n! \geq \sum_{j=1}^{m} \binom{n}{|A_{j1}| + \cdots + |A_{jp}|} |A_{j1}|! \cdots |A_{jp}|! \big(n - |A_{j1}| - \cdots - |A_{jp}|\big)!$$

$$= \sum_{j=1}^{m} \frac{n!}{\binom{|A_{j1}|+\cdots+|A_{jp}|}{|A_{j1}|,\dots,|A_{jp}|}}.$$

The theorem follows. ∎

To deduce Theorem 1.5 from Theorem 2.5, we use the following lemma, which originally appeared in somewhat different and incomplete form in [10], used there to prove Erdős's Theorem 1.2 by means of Theorem 2.2, and appeared in complete form in [7, Lemma 3.1.3]. We give a very short proof, which seems to be new.

Lemma 3.1 (Harper–Klain–Rota). *Suppose* $M_1, \dots, M_N \in \mathbb{R}$ *satisfy* $M_1 \geq M_2 \geq \cdots \geq M_N \geq 0$, *and let* R *be an integer with* $1 \leq R \leq N$. *If* $q_1, \dots, q_N \in [0,1]$ *have sum*

$$q_1 + \cdots + q_N \leq R,$$

then

$$q_1 M_1 + \cdots + q_N M_N \leq M_1 + \cdots + M_R.$$

Proof. By assumption,

$$\sum_{k=R+1}^{N} q_k \leq \sum_{k=1}^{R} (1 - q_k).$$

Hence, by the condition on the M_k,

$$\sum_{k=R+1}^{N} q_k M_k \leq M_R \sum_{k=R+1}^{N} q_k \leq M_R \sum_{k=1}^{R} (1 - q_k) \leq \sum_{k=1}^{R} (1 - q_k) M_k,$$

which is equivalent to the conclusion. ∎

Proof of Theorem 1.5. Let S be any finite set that contains all A_{jk}. Write down the LYM inequality from Theorem 2.5.

From the m weak partial compositions $(A_{j1}, \dots, A_{jp})$ of S, collect those whose shape is $(a_1, \dots, a_p)$ into the set $C(a_1, \dots, a_p)$. Label the p-multinomial coefficients for integers $n' \leq n$ as $M'_1, M'_2, \dots$ so that $M'_1 \geq M'_2 \geq \cdots$. If M'_k is $\binom{n'}{a_1, \dots, a_p}$, let $q'_k := |C(a_1, \dots, a_p)| / M'_k$. By Theorem 2.5, the q'_k's and M'_k's satisfy all the conditions of Lemma 3.1 with N replaced by the number of p-tuples $(a_1, \dots, a_p)$ whose sum is at most n, that is $\binom{n+p}{p}$, and R replaced by $\min(N, r^p)$. Hence

$$\sum_{a_1 + \cdots + a_p \leq n} |C(a_1, \dots, a_p)| \leq M'_1 + \cdots + M'_R.$$

The conclusion of the theorem now follows, since

$$m = \sum_{a_1 + \cdots + a_p \leq n} |C(a_1, \dots, a_p)|. \qquad \blacksquare$$

4. Consequences

As promised in Section 1, we now state special cases of Theorems 1.5/2.5 that unify pairs of Theorems 1.2, 1.3, and 1.4 as well as their LYM companions.

The first special case unifies Theorems 1.2/2.2 and 1.3/2.3. (It is a corollary of the proof of the main theorems, not of the theorems themselves. See [2] for a very short, direct proof.)

Corollary 4.1. *Suppose* $(A_{j1}, \dots, A_{jp})$ *are* m *different weak compositions of* S *into* p *parts such that for each* $k \in [p-1]$, *the set* $\{A_{jk} : 1 \leq j \leq m\}$ *is* r*-chain-free. Then*

$$\sum_{j=1}^{m} \frac{1}{\binom{n}{|A_{j1}|,\dots,|A_{jp}|}} \leq r^{p-1}.$$

Consequently, m *is bounded by the sum of the* r^{p-1} *largest* p*-multinomial coefficients for* n.

Proof. We note that, for a family of m weak compositions of S, the condition of Theorem 2.5 for a particular $k \in [p-1]$ is equivalent to $\{A_{jk}\}_j$ being r-chain-free. Thus by the hypothesis of the corollary, the hypothesis of the theorem is met for $k = 1, \dots, p-1$. Then the proof of Theorem 2.5 goes through perfectly with the only difference, explained in the proof, that (even without a condition on $k = p$) we obtain $N \leq r^{p-1}$. In the proof of Theorem 1.5, under our hypotheses the sets $C(a_1, \dots, a_p)$ with $a_1 + \cdots + a_p < n$ are empty. Therefore we take only the p-multinomial coefficients for n, labelled $M_1 \geq M_2 \geq \cdots$. In applying Lemma 3.1 we take $R = \min(N, r^{p-1})$ and summations over $a_1 + \cdots + a_p = n$. With these alterations the proof fits Corollary 4.1. ∎

A good way to think of Corollary 4.1 is as a theorem about partial weak compositions, obtained by dropping the last part from each of the weak compositions in the corollary.

Corollary 4.2. *Fix* $p \geq 2$ *and* $r \geq 1$. *Suppose* $(A_{j1}, \dots, A_{jp})$ *are* m *different weak partial compositions of an* n*-set* S *into* p *parts such that for each* $k \in [p]$, *the set* $\{A_{jk} : 1 \leq j \leq m\}$ *is* r*-chain-free. Then* m *is bounded by the sum of the* r^p *largest* $(p+1)$*-multinomial coefficients for* n. ∎

A difference between this and Theorem 1.5 is that Corollary 4.2 has a weaker and simpler hypothesis but a much weaker bound. But the biggest difference is the omission of an accompanying LYM inequality. Corollary 4.1 obviously implies one, but it is weaker than that in Theorem 2.5 because, since the top number in the latter can be less than n, the denominators are much smaller. We do not present in Corollary 4.2 an LYM inequality of the kind in Theorem 2.5 for the very good reason that none is possible; that is the meaning of Example 2.1.

The second specialization constitutes a weak common refinement of Theorems 1.2/2.2 and 1.4/2.4. We call it weak because its specialization to the case $B_j = S - A_j$, which is the situation of Theorems 1.2/2.2, is weaker than those theorems.

Corollary 4.3. *Let r be a positive integer. Suppose (A_j, B_j) are m pairs of sets such that $A_j \cap B_j = \varnothing$ and, for all $I \subseteq [m]$ with $|I| = r+1$, there exist distinct $i, j \in I$ for which $A_j \cap B_k \neq \varnothing \neq A_k \cap B_j$. Let $n = \max_j \left(|A_j| + |B_j|\right)$. Then*

$$\sum_{j=1}^{m} \frac{1}{\binom{|A_j|+|B_j|}{|A_j|}} \leq r.$$

Consequently, m is bounded by the sum of the r largest binomial coefficients $\binom{n'}{k}$ for $0 \leq k \leq n' \leq n$. This bound can be attained for all n and r.

Proof. Set $p = 2$ in Theorems 1.5/2.5. To attain the bound, let A_j range over all k-subsets of $[n]$ and let $B_j = [n] - A_j$. ■

The last special case of Theorems 1.5/2.5 we would like to mention is that in which $r = 1$; it unifies Theorems 1.3/2.3 and 1.4/2.4.

Corollary 4.4. *Suppose $(A_{j1}, \ldots, A_{jp})$ are m different weak set compositions into p parts with the condition that, for all $k \in [p]$ and all distinct $i, j \in [m]$, either $A_{ik} = A_{jk}$ or*

$$A_{ik} \cap \bigcup_{l \neq k} A_{jl} \neq \varnothing \neq A_{jk} \cap \bigcup_{l \neq k} A_{il}.$$

and let $n \geq \max_j \left(|A_{j1}| + \cdots + |A_{jp}|\right)$. Then

$$\sum_{j=1}^{m} \frac{1}{\binom{|A_{j1}|+\cdots+|A_{jp}|}{|A_{j1}|,\ldots,|A_{jp}|}} \leq 1.$$

Consequently, m is bounded by the largest p-multinomial coefficient for n. The bound can be attained for every n and p.

Proof. Everything follows from Theorems 1.5/2.5 except the attainability of the upper bound, which is a consequence of Theorem 1.3. ■

5. The maximum number of compositions

Although the bounds in all the previously known Sperner generalizations of Section 1 can be attained, for the most part that seems not to be the case in Theorem 1.5. The key difficulty appears in the combination of r-families with compositions as in Corollary 4.1. (We think it makes no difference if we allow partial compositions but we have not proved it.) We begin with a refinement of Lemma 3.1. A weak set composition has **shape** $(a_1, \ldots, a_p)$ if $|A_k| = a_k$ for all k.

Lemma 5.1. *Given values of n, r, and p such that $r^{p-1} \leq \binom{n+p-1}{p-1}$, the bound in Corollary 4.1 can be attained only by taking all weak compositions of shape $(a_1, \ldots, a_p)$ that give p-multinomial coefficient larger than the $(r^{p-1}+1)$-st largest such coefficient $M_{r^{p-1}+1}$, and none whose shape gives a smaller coefficient than the (r^{p-1})-st largest such coefficient $M_{r^{p-1}}$.*

Proof. First we need to characterized sharpness in Lemma 3.1. Our lemma is a slight improvement on [7, Lemma 3.1.3].

Lemma 5.2. *In Lemma 3.1, suppose that $M_R > 0$. Then there is equality in the conclusion if and only if*

$$q_k = 1 \text{ if } M_k > M_R \qquad \text{and} \qquad q_k = 0 \text{ if } M_k < M_R$$

and also, letting $M_{R'+1}$ and $M_{R''}$ be the first and last M_k's equal to M_R,

$$q_{R'+1} + \cdots + q_{R''} = R - R'. \qquad ■$$

In Lemma 5.1, all $M_k > 0$ for $k \leq \binom{n+p-1}{p-1}$. (We assume N is no larger than $\binom{n+p-1}{p-1}$. The contrary case is easily derived from that one.) It is clear that, when applying Lemma 3.1, we have to have in our set of weak compositions all those of the shapes $(a_1, \ldots, a_p)$ for which $\binom{n}{a_1,\ldots,a_p} > M_{r^{p-1}}$ and none for which $\binom{n}{a_1,\ldots,a_p} < M_{r^{p-1}}$. The rest of the m weak compositions can have any shapes for which $\binom{n}{a_1,\ldots,a_p} = M_{r^{p-1}}$. If $M_{r^{p-1}} > M_{r^{p-1}+1}$ this means we must have all weak compositions with shapes for which $\binom{n}{a_1,\ldots,a_p} > M_{r^{p-1}+1}$. ■

To explain why the bound cannot usually be attained, we need to define the "first appearance" of a size a_i in the descending order of p-multinomial coefficients for n.

Fix $p \geq 3$ and n and let $n = \nu p + \rho$ where $0 \leq \rho < p$. In $\binom{n}{a_1,\ldots,a_p}$, the a_i are the **sizes**. The multiset of sizes is the **form** of the coefficient. Arrange the multinomial coefficients in weakly decreasing order: $M_1 \geq M_2 \geq M_3 \geq \cdots$. (There are many such orderings; choose one arbitrarily, fix it, and call it **the descending order** of coefficients.) Thus, for example,

$$M_1 = \binom{n}{\nu,\ldots,\nu} > M_2 = \binom{n}{\nu+1,\nu,\ldots,\nu,\nu-1}$$

$$= M_3 = \cdots = M_{p(p-1)+1} \quad \text{if} \quad p \mid n$$

since $M_3, \ldots, M_{p(p-1)+1}$ have the same form as M_2, and

$$M_1 = \binom{n}{\nu+1,\ldots,\nu} = \cdots = M_{\binom{p}{\rho}} > M_{\binom{p}{\rho}+1} \quad \text{if} \quad p \nmid n,$$

where the form of M_1 has ρ sizes equal to $\nu + 1$, so $M_1, \ldots, M_{\binom{p}{\rho}}$ all have the same form.

As we scan the descending order of multinomial coefficients, each possible size κ, $0 \leq \kappa \leq n$, appears first in a certain M_i. We call M_i the **first appearance** of κ and label it L_κ. For example, if $p \mid n$, $L_\nu = M_1 > L_{\nu+1} = L_{\nu-1} = M_2$, while if $p \nmid n$ then $L_\nu = L_{\nu+1} = M_1$. It is clear that $L_\nu > L_{\nu-1} > \ldots$ and $L_{\nu+1} > L_{\nu+2} > \ldots$, but the way in which the lower L_κ's, where $\kappa \leq \nu$, interleave the upper ones is not obvious. We write L_k^* for the k-th L_κ in the descending order of multinomial coefficients. Thus $L_1^* = L_\nu$; $L_2^* = L_{\nu+1}$ and $L_3^* = L_{\nu-1}$ (or vice versa) if $p \mid n$, and $L_2^* = L_{\nu+1}$ if $p \nmid n$ while $L_3^* = L_{\nu+2}$ or $L_{\nu-1}$.

Theorem 5.1. *Given $r \geq 2$, $p \geq 3$, and $n \geq p$, the bound in Corollary 4.1 cannot be attained if $L_r^* > M_{r^{p-1}+1}$.*

The proof depends on the following lemma.

Lemma 5.3. *Let $r \geq 2$ and $p \geq 3$, and let $\kappa_1, \ldots, \kappa_r$ be the first r sizes that appear in the descending order of p-multinomial coefficients for n. The number of all coefficients with sizes drawn from $\kappa_1, \ldots, \kappa_r$ is less than r^{p-1} and their sum is less than $M_1 + \cdots + M_{r^{p-1}}$.*

Proof. Clearly, $\kappa_1, \ldots, \kappa_r$ form a consecutive set that includes ν. Let κ be the smallest and κ' the largest. One can verify that, in $\binom{n}{\kappa,\ldots,\kappa,x}$ and $\binom{n}{\kappa',\ldots,\kappa',y}$, it is impossible for both x and y to lie in the interval $[\kappa, \kappa']$ as long as $(r-1)(p-2) > 0$. ∎

Proof of Theorem 5.1. Suppose the upper bound of Corollary 4.1 is attained by a certain set of weak compositions of S, an n-element set. For each of the first r sizes $\kappa_1, \ldots, \kappa_r$ that appear in the descending order of p-multinomial coefficients, L_{κ_i} has sizes drawn from $\kappa_1, \ldots, \kappa_r$ and at least one size κ_i. Taking all coefficients M_k that have the same forms as the L_{κ_i}, κ_i will appear in each position j in some M_k. By hypothesis and Lemma 5.1, among our set of weak compositions, every κ_i-subset of S appears in every position in the weak compositions. If any subset of S of a different size from $\kappa_1, \ldots, \kappa_r$ appeared in any position, there would be a chain of length r in that position. Therefore we can only have weak compositions whose sizes are among the first r sizes. By Lemma 5.3, there are not enough of these to attain the upper bound. ∎

Theorem 5.1 can be hard to apply because we do not know $M_{r^{p-1}+1}$. On the other hand, we do know L_κ since it equals $\binom{n}{\kappa,a_2,\ldots,a_p}$ where $a_2, \ldots, a_p$ are as nearly equal as possible. A more practical criterion for nonattainment of the upper bound is therefore

Corollary 5.1. *Given $r \geq 2$, $p \geq 3$, and $n \geq p$, the bound in Corollary 4.1 cannot be attained if $L_r^* > L_{r+1}^*$.*

Proof. It follows from Lemma 5.3 that L_{r+1}^* is one of the first r^{p-1} coefficients. Thus $L_r^* > L_{r+1}^* \geq M_{r^{p-1}+1}$ and Theorem 5.1 applies. ∎

It seems clear that L_r^* will almost always be larger than L_{r+1}^* (if $r \geq 3$ or $p \nmid n$) so our bound will not be attained. However, cases of equality do exist. For instance, take $p = 3$, $r = 3$, and $n = 10$; then $L_5^* = L_1 = \binom{10}{5,4,1} = 1260$ and $L_6^* = L_6 = \binom{10}{6,2,2} = 1260$. Thus if $r = 5$, Corollary 5.1 does not apply here. (We think the bound is still not attained but we cannot prove it.) We can isolate the instances of equality for each r, but as r grows larger the calculations quickly become extensive. Thus we state the results only for small values of r.

Proposition 5.1. *The bound in Corollary* 4.1 *cannot be attained if* $2 \leq r \leq 5$ *and* $p \geq 3$ *and* $n \geq r-1$, *except possibly when* $r = 2$, $p \mid n$, *and* $p = 3, 4, 5$, *or when* $r = 4$, $p \geq 4$, *and* $n = 2p-1$, *or when* $r = 5$, $p = 3$, *and* $n = 10$.

Proof sketch. Suppose $p \nmid n$. We have verified (by long but routine calculations which we omit) that $L_1^* = L_2^* > L_3^* > L_4^* > L_5^* > L_6^*$ except that $L_4^* = L_5^*$ if $\rho = p-1$ and $p \geq 4$ and $\nu = 1$ and $L_5^* = L_6^*$ when $p = \nu = 3$ and $\rho = 1$.

If $p \mid n$ then $L_1^* > L_2^* = L_3^* > L_4^* > L_5^* > L_6^*$. This implies the proposition for $r = 3$, 4, or 5. We approach $r = 2$ differently. The largest coefficients are

$$M_1 = \binom{n}{\nu, \ldots, \nu} > M_2 = \binom{n}{\nu+1, \nu, \ldots, \nu, \nu-1} = \cdots$$
$$= M_{p(p-1)+1} > M_{p(p-1)+2}.$$

If $p(p-1)+1 \leq r^{p-1}$, the bound is unattainable by Theorem 5.1. That is the case when $p \geq 6$.

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A Quick Proof of Sprindzhuk's Decomposition Theorem

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Dedicated to the memory of V. G. Sprindzhuk

In [11] Sprindzhuk proved the following striking theorem.

Theorem 1 (Sprindzhuk [11]). *Let $F(x,y) \in \mathbb{Q}[x,y]$ be a $\mathbb{Q}$-irreducible polynomial satisfying*

$$F(0,0) = 0, \quad \frac{\partial F}{\partial y}(0,0) \neq 0. \tag{1}$$

Then for all but finitely many prime numbers p, the polynomial $F(p,y)$ is $\mathbb{Q}$-irreducible.

Actually, prime numbers can be replaced by prime powers, as well as by numbers of the form $1/t$, where $t \in \mathbb{Z}$, $t \neq 0$: see Corollary 3.

In the subsequent paper [12] (see also [13, 14] for a more detailed exposition) Sprindzhuk obtained an even more amusing result. To formulate it, recall that the *height* of a rational number $\alpha = a/b$ (where a and b are coprime integers) is defined by

$$H(\alpha) = \max\{|a|, |b|\}. \tag{2}$$

One immediately verifies that

$$H(\alpha) = \prod_{v \in M_{\mathbb{Q}}} \max\{1, |\alpha|_v\} = \prod_{v \in M_{\mathbb{Q}}} \max\{1, |\alpha|_v^{-1}\}, \tag{3}$$

where $M_{\mathbb{Q}}$ is the set of all places of the field $\mathbb{Q}$ (that is, $M_{\mathbb{Q}} = \{\text{primes}\} \cup \{\infty\}$).

For $\alpha \in \mathbb{Q}$ put $V(\alpha) = \{v \in M_{\mathbb{Q}} : |\alpha|_v < 1\}$.

Theorem 2 (Sprindzhuk [12]). *Let $F(x,y)$ be as in Theorem 1 and ε a positive number. For every $\alpha \in \mathbb{Q}$ let $d_1(\alpha),\dots,d_k(\alpha)$ be the degrees of the $\mathbb{Q}$-irreducible factors of $F(\alpha,y)$ (so that $d_1(\alpha)+\dots+d_k(\alpha)=\deg_y F$). Then for all but finitely many $\alpha \in \mathbb{Q}$ there is a partition $V(\alpha)=V_1\cup\ldots\cup V_k$ such that*

$$\left|\frac{-\sum_{v\in V_i}\log|\alpha|_v}{\log H(\alpha)}-\frac{d_i(\alpha)}{\deg_y F}\right|<\varepsilon \qquad (i=1,\dots,k). \tag{4}$$

We do not formally assert that the partition sets $V_1,\dots,V_k$ are non-empty. However, (4) implies that they are indeed non-empty when ε is sufficiently small (in fact, when $\varepsilon<1/\deg_y F$).

Theorem 1 easily follows from Theorem 2. Put $\Omega=\{\text{prime powers}\}\cup\{1/t : t\in\mathbb{Z}, |t|>1\}$.

Corollary 3. *Let $F(x,y)$ be as in Theorem 1. Then $F(\omega,y)$ is $\mathbb{Q}$-irreducible for all but finitely many $\omega\in\Omega$.*

Proof. As we observed above, the partition sets $V_1,\dots,V_k$ are non-empty when ε is sufficiently small. But for every $\omega\in\Omega$ the set $V(\omega)$ consists of a single element, and cannot be partitioned into more than one non-empty part. ■

Here is another amazing consequence of Theorem 2 (the proof is immediate).

Corollary 4. *Let $F(x,y)$ be as in Theorem 1 and let $\{q_\nu\}$, $\{r_\nu\}$ be two sequences of prime powers such that $\lim_{\nu\to\infty}\log q_\nu/\log r_\nu$ exists and is irrational. Then $F(q_\nu r_\nu,y)$ is $\mathbb{Q}$-irreducible for all but finitely many ν.* ■

We invite the reader to invent many other corollaries of this wonderful theorem.

Actually Sprindzhuk in [12] obtained a yet sharper version of Theorem 2 with ε replaced by an error term of order $\big(\log H(\alpha)\big)^{-1/2}$. To prove this he used Siegel's Lemma and some sophisticated machinery from the theory of Diophantine approximation and transcendence such as the cancellation of factorials and a zero estimate (Lemma 6 of [11]). He also used Eisenstein's theorem, which is easy when (1) is assumed.

In the final paragraph of the Russian edition of his book [13], Sprindzhuk wrote that, while methods of Diophantine approximation are used in the proof of Theorem 2, its formulation

> "...involves no concepts related to the theory of Diophantine approximation. This gives hope that a different proof exists, which is independent of the theory of Diophantine approximation."

Indeed, such a proof was soon after found by Bombieri [1], who used the machinery of Weil functions and Néron-Tate height. Weil functions were also employed by Fried [9] in the prime-power case. It was Bombieri who pointed out the connection with G-functions and Fuchsian differential operators of arithmetic type. This connection was further developed by Dèbes (and Zannier) [3, 4, 5, 6].

The object of the present note is to point out that Theorem 2 itself can be established rather quickly, also along the lines of Sprindzhuk's original articles, but without most of the sophisticated machinery. Our proof relies only on the simplest properties of heights (see Proposition 5 below) and Eisenstein's theorem.

Recall the definition of the height of an algebraic number. This is

$$H(\alpha) = \left(\prod_{v \in M_K} \max\left\{1, |\alpha|_v^{[K_v:\mathbb{Q}_v]}\right\} \right)^{1/[K:\mathbb{Q}]}, \tag{5}$$

where K is a number field containing α and M_K is the set of valuations on K, which are normalized to extend the standard valuations of $\mathbb{Q}$. As usual, K_v and $\mathbb{Q}_v$ stand for the topological completions with respect to $v \in M_K$.

It is straightforward to verify that the right-hand side of (5) does not depend on the choice of the field K. Also (3) implies that this definition is compatible with the definition of the height of a rational number from (2).

The product formula

$$\prod_{v \in M_K} |\alpha|_v^{[K_v:\mathbb{Q}_v]} = 1 \qquad (\alpha \in K^*)$$

implies that for any $V \subset M_K$ and $\alpha \in K^*$ one has the following "Liouville inequality":

$$\prod_{v \in V} |\alpha|_v^{[K_v:\mathbb{Q}_v]} \geq H(\alpha)^{-[K:\mathbb{Q}]}. \tag{6}$$

The following two well-known properties of the height function are (almost) immediate consequences of its definition (5).

Proposition 5. *Let α, β be algebraic numbers and $F(x,y)$ a polynomial with algebraic coefficients. Put $m = \deg_x F$ and $n = \deg_y F$.*

1. *For $\gamma = F(\alpha,\beta)$ one has $H(\gamma) \ll H(\alpha)^m H(\beta)^n$.*

2. *Assume that F is not divisible by $x-\alpha$. Then $F(\alpha,\beta)=0$ implies that $H(\beta) \ll H(\alpha)^m$.*

Constants implied by "$\ll$" depend only on the polynomial F.

Proof. Part "1" is straightforward. To prove "2", write $F(x,y) = f_n(x)y^n + \cdots + f_0(y)$. By the assumption, not all of the numbers $f_0(\alpha),\ldots,f_n(\alpha)$ vanish. Put $\nu = \max\left\{ j \,:\, f_j(\alpha) \neq 0 \right\}$.

Let K be a number field containing α, β and the coefficients of F. The equality $f_\nu(\alpha)\beta^\nu + f_{\nu-1}(\alpha)\beta^{\nu-1} + \cdots + f_0(\alpha) = 0$ implies that

$$\max\left\{1, |\beta|_v\right\} \le \max\left\{1, |\nu|_v\right\}$$
$$\max\left\{1, \left|f_{\nu-1}(\alpha)/f_\nu(\alpha)\right|_v, \ldots, \left|f_0(\alpha)/f_\nu(\alpha)\right|_v\right\} \qquad (v \in M_K).$$

Using the product formula, we obtain

$$H(\beta) \le H(\nu)\Bigg(\prod_{v\in M_K} \max\left\{1, \left|f_{\nu-1}(\alpha)/f_\nu(\alpha)\right|_v, \ldots,\right.$$
$$\left.\left|f_0(\alpha)/f_\nu(\alpha)\right|_v\right\}^{[K_v:\mathbb{Q}_v]}\Bigg)^{1/[K:\mathbb{Q}]}$$
$$= \nu\Bigg(\prod_{v\in M_K} \max\left\{\left|f_\nu(\alpha)\right|_v, \left|f_{\nu-1}(\alpha)\right|_v, \ldots,\right.$$
$$\left.\left|f_0(\alpha)\right|_v\right\}^{[K_v:\mathbb{Q}_v]}\Bigg)^{1/[K:\mathbb{Q}]} \ll H(\alpha)^m,$$

as wanted. ■

Recall also Eisenstein's theorem.

Theorem 6. *Let $Y(x) = a_0 + a_1x + a_2x^2 + \cdots$ be a power series with coefficients in a number field K, algebraic over the field $K(x)$. Then for* *every* $v \in M_K$ *there exists* $c_v \geq 1$ *such that all but finitely many* c_v *are equal to* 1, *and*

$$|a_j|_v \leq c_v^j \qquad (v \in M_K, \quad j = 1, 2, \ldots). \tag{7}$$

Classically, Eisenstein's theorem reads as follows: there exists a positive integer T such that $T^j a_j$ are algebraic integers for $j = 1, 2, \ldots$. This immediately implies Theorem 6. Indeed, for non-archimedean v one may put $c_v = |T|_v^{-1}$. For archimedean v, the existence of c_v follows from the fact that the convergence radius of a complex algebraic power series is positive.

Eisenstein's theorem goes back to Eisenstein's paper [8]. See [10, page 151] for an old-fashioned proof and [7] for a modern quantitative argument. See also [2, page 28] for an especially quick proof when $K = \mathbb{Q}$, which suffices for the present note. In addition, if $a_0 = 0$ and $F\big(x, Y(x)\big) = 0$, where $F(x, y) \in \mathbb{Z}[x, y]$ satisfies (1), then a very easy induction gives the value $T = \big(\partial F/\partial y(0,0)\big)^2$, and in fact this case suffices as well.

Proof of Theorem 2. Put $m = \deg_x F$ and $n = \deg_y F$. To prove the theorem, it is sufficient to find a partition $V(\alpha) = V_1 \cup \ldots \cup V_k$ satisfying

$$\frac{-\sum_{v \in V_i} \log |\alpha|_v}{\log H(\alpha)} \leq \frac{d_i(\alpha)}{n} + \varepsilon \qquad (i = 1, \ldots, k). \tag{8}$$

Indeed, by the second equality in (3),

$$\sum_{i=1}^{k} \frac{-\sum_{v \in V_i} \log |\alpha|_v}{\log H(\alpha)} = 1 = \sum_{i=1}^{k} \frac{d_i(\alpha)}{n}.$$

Hence (8) implies that

$$\frac{-\sum_{v \in V_i} \log |\alpha|_v}{\log H(\alpha)} \geq \frac{d_i(\alpha)}{n} - (k-1)\varepsilon \qquad (i = 1, \ldots, k),$$

and (4) follows after redefining ε.

It follows from (1) that there exists a power series $Y(x) = a_1 x + a_2 x^2 + \cdots$ with rational coefficients satisfying $F\big(x, Y(x)\big) = 0$. Put

$$N = \lceil 4m(n-1)/\varepsilon \rceil. \tag{9}$$

There is a non-zero polynomial $G(x, y) \in \mathbb{Q}[x, y]$ satisfying

$$\deg_y G \leq n - 1, \quad \deg_x G \leq N, \tag{10}$$

$$\operatorname{ord}_{x=0} G\big(x, Y(x)\big) \geq nN. \tag{11}$$

(Indeed, the vector space of polynomials satisfying (10) is of dimension $n(N+1)$, while (11) is equivalent to nN linear relations.) In the sequel, constants implied by "$O(\cdot)$", "$\ll$" and "$\gg$" may depend only on F, G and ε.

Put $U(x) = G\bigl(x, Y(x)\bigr)$. By Eisenstein's theorem, for every $v \in M_{\mathbb{Q}}$ there exists $c_v \geq 1$ such that all but finitely many c_v are equal to 1, and the coefficients of the power series $Y(x) = \sum_{j=1}^{\infty} a_j x^j$ and $U(x) = \sum_{j=nN}^{\infty} b_j x^j$ satisfy

$$|a_j|_v, |b_j|_v \leq c_v^j \qquad (v \in M_{\mathbb{Q}}). \tag{12}$$

For $\alpha \in \mathbb{Q}$ put

$$V'(\alpha) = \left\{ v \in V(\alpha) \,:\, |\alpha|_v \begin{array}{ll} \leq 1/(2c_v) & \text{if } v = \infty, \\ < 1/c_v & \text{if } v < \infty\,. \end{array} \right\}, \quad V''(\alpha) = V(\alpha) \setminus V'(\alpha).$$

Since $-\sum_{v \in V''(\alpha)} \log |\alpha|_v \ll 1$, for all but finitely many α we have

$$\frac{-\sum_{v \in V''(\alpha)} \log |\alpha|_v}{\log H(\alpha)} \leq \frac{\varepsilon}{2}.$$

Hence it is sufficient to find a partition $V'(\alpha) = V_1' \cup \ldots \cup V_k'$ such that

$$\frac{-\sum_{v \in V_i'} \log |\alpha|_v}{\log H(\alpha)} \leq \frac{d_i(\alpha)}{n} + \frac{\varepsilon}{2} \qquad (i = 1, \ldots, k), \tag{13}$$

for then putting, say, $V_1 = V_1' \cup V''(\alpha)$ and $V_i = V_i'$ for $i \geq 2$, we obtain (8).

Thus, fix $\alpha \in \mathbb{Q}$ and let $F(\alpha, y) = f_1(y) \cdots f_k(y)$ be the decomposition of $F(\alpha, y)$ into $\mathbb{Q}$-irreducible factors. We may assume (discarding finitely many α at which the y-discriminant of $F(x, y)$ vanishes) that the polynomials f_i are pairwise coprime. We put $d_i = \deg f_i$.

For any $v \in V'(\alpha)$ the series $Y(x)$ converges v-adically at α. Its sum in $\mathbb{Q}_v$, denoted by $Y_v(\alpha)$, is a zero of $F(\alpha, y)$. Define the partition $V'(\alpha) = V_1' \cup \ldots \cup V_k'$ as follows:

$$V_i' = \bigl\{ v \in V'(\alpha) \,:\, Y_v(\alpha) \text{ is a zero of } f_i(y) \bigr\} \qquad (i = 1, \ldots, k).$$

Now fix i and let $\beta = \beta_i$ be a zero of $f_i(y)$. Again discarding finitely many α, we may assume that $\eta := G(\alpha, \beta) \neq 0$. Indeed, since $F(x, y)$ is irreducible, and $\deg_y G < \deg_y F$, the system of algebraic equations $F(\alpha, \beta) = G(\alpha, \beta) = 0$ has only finitely many solutions.