# Pedagogy and Content in Middle and High School Mathematics

G. Donald Allen and
Amanda Ross (Eds.)

SensePublishers

# Pedagogy and Content in Middle and High School Mathematics

# Pedagogy and Content in Middle and High School Mathematics

*Edited by*

**G. Donald Allen**
*Texas A&M University, USA*

and

**Amanda Ross**
*A. A. Ross Consulting and Research, USA*

SENSE PUBLISHERS
ROTTERDAM/BOSTON/TAIPEI

A C.I.P. record for this book is available from the Library of Congress.

ISBN: 978-94-6351-135-3 (paperback)
ISBN: 978-94-6351-136-0 (hardback)
ISBN: 978-94-6351-137-7 (e-book)

Published by: Sense Publishers,
P.O. Box 21858,
3001 AW Rotterdam,
The Netherlands
https://www.sensepublishers.com/

*Printed on acid-free paper*

# TABLE OF CONTENTS

# PREFACE

This book provides articles, included in *Focus on Mathematics Pedagogy and Content*, a newsletter for middle and high school teachers, published by Texas A&M University. The book covers all five NCTM content strands, focusing only on grades 6–12. The articles may be used as a reference for teachers, on both effective ways to teach mathematics, as well as mathematics content knowledge.

High level mathematics content and problem solving processes are presented in different ways, including via historical information and creative real-world contexts. This book offers historical perspectives and connections, which are not typically found in other books that examine instructional strategies for various mathematics topics. The book will benefit those readers, who desire to learn more about the history of mathematics and its connection to teaching in the mathematics classroom. As related to problem solving, many articles present different ways of representing mathematics content, ways of connecting these representations, and different ways to approach the same type of problem. In addition, student misconceptions are interspersed throughout the book.

The book also briefly delves into assessments, looking at an amalgamation of topics, related to formative and summative assessments. These articles focus on test construction, viewpoints, background, and types of assessments. Finally, a whole section on "Teaching Tips" is included in the book, in addition to a section on games and technology integration.

We would like to thank all of the contributors to this book. All of the contributions were guided, based upon personal interest. Articles were not submitted, in response to a particular call for articles or particular content domain request. Authors were given complete autonomy in deciding on article content. This lack of structure resulted in a wide variety of articles on many different mathematics content and pedagogy topics, which certainly added to the uniqueness of this book.

We hope you enjoy the book!

# PART I

# CONTENT AND PEDAGOGY

# SECTION 1

# NUMBER

G. DONALD ALLEN

# 1. A BRIEF HISTORY OF ZERO

Zero is a relatively recent addition to mathematics. Indeed, entire civilizations lasting longer than our entire western era (i.e. more than two thousand years) flourished and perished having built the pyramids and the wonders of the world, without any notion of zero. Zero, which is taught to youngsters, is such an important concept, and like much of mathematics, was invented out of necessity.

One of our principal uses of zero is as a placeholder in our system of enumeration. How else could we write 2005 without the zero? The ancient Egyptians, Babylonians, Greeks, and Romans all knew how to do so. Placeholders are a mere convenience of our enumeration, not an essential part of enumeration. Systems of enumeration are shown in Figure 1.

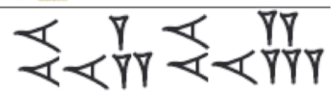
| | |
|---|---|
| Egyptian (hieroglyphs) |  |
| Babylonian (cuneiform) |  |
| Greek (alpha-numerical) | ‘βε |
| Roman (alpha-numerical) | **MMV** |

*Figure 1. Systems of enumeration*

However, they do help with our algorithms of calculation such as addition, subtraction, multiplication, and division. For example, our modern division algorithm is about five hundred years old, and once was an advanced subject taught only in Italy. These algorithms take keen advantage of our zero placeholder, and make rapid hand calculations possible. More important, they make the realization of truly large numbers such as a google, $10^{100}$, possible.

The other use of zero is as a number itself. You can be the judge of which is more important. But for basic business type mathematics calculations, it would be the place holding value that is probably greatest. In higher math, the actual value of zero is extremely important.

While the Babylonians and ancient Greeks did finally evolve to a symbolic placeholder for zero, it was not really a number. What we do know is that by around

*G. D. Allen & A. Ross (Eds.), Pedagogy and Content in Middle and High School Mathematics, 5–6.*

650 AD, the use of zero as a number came into Indian mathematics. Its original form is very much like our own zero, 0, only a little smaller, though there is evidence also that a single dot, . , was used to denote an empty space. Links: History of zero: http://www-groups.dcs.st-and.ac.uk/~history/HistTopics/Zero.html Enumeration: chapters 3 and 4 of http://www.math.tamu.edu/%7Edallen/masters/hist_frame.htm

G. DONALD ALLEN

# 2. APPROXIMATING PI

Ever since mankind began surveying areas and building, the need to measure circles has been important. For this task, we need $\pi$. As we know today, $\pi$ is one of the most peculiar of numbers. It is not rational, but instead, irrational, and is a special kind of irrational number, called a transcendental number, meaning that it cannot be the solution of a polynomial equation with integer coefficients. Indeed, it was only in 1840 that such numbers were even found, and this was thousands of years after the ancients first mused on what $\pi$ might be. Well, everyone knew $\pi$ was a little bit larger than three, but to achieve an accurate approximation was an elusive task. Let's look at a couple of methods and approximations from various civilizations.

## ANCIENT EGYPTIANS

"A square of side 8 has the area of a circle of diameter 9." The area of any circle was then approximated using proportion using the formula and what was understood to be an area to square of the radius formula.

$$\frac{A}{r^2} = \frac{64}{\left(\frac{9}{2}\right)^2} = \frac{256}{81}$$

The form we use today is shown below.

$$A = \frac{256}{81} r^2$$

Cut off each corner of the square of side length 9 divided horizontally and vertically in thirds, as shown, and add the resulting five squares of radius three and four triangles of half that size to get $5(3)^2 + 4\left(\frac{1}{2} \cdot 3^2\right) = 63 \approx 64$. See Figure 1.

*G. D. Allen & A. Ross (Eds.), Pedagogy and Content in Middle and High School Mathematics, 7–10.*

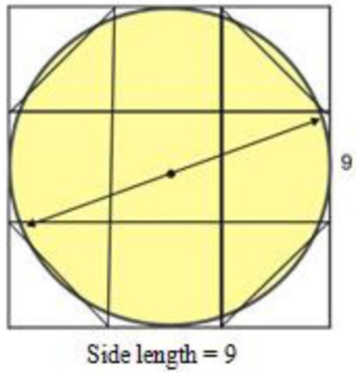



*Figure 1. Approximation by Ancient Egyptians*

## ANCIENT GREEKS

$\pi \approx \frac{22}{7}$. This incredibly remarkable formula was determined by Archimedes, the greatest of the ancient mathematicians. He was able to determine the areas of inscribed and circumscribed regular polygons of 6, 12, 24, 48, and finally 96 sides. In this way, he found lower and upper estimates of $\pi$, the lower estimate being $3\frac{10}{71}$. See Figure 2, where polygons of up to 24 sides are shown. What Archimedes did was discover a very clever relation between the areas of these figures, as the number of sides increased. This allowed the prodigious computational feat. The formulas of Archimedes were used even until modern times to compute ever more accurate approximations to $\pi$.

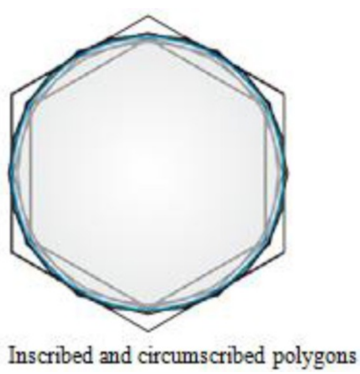



*Figure 2. Approximation by Ancient Greeks*

## ANCIENT CHINESE

While it is uncertain how the computation was made, the Chinese of the 5th century gave us the purely elementary fraction approximation to $\pi$ given by

$$\frac{355}{113} = 3.14159292$$

Now to ten places $\pi = 3.141592654$. So you can see that the approximation, $\pi - \frac{355}{113} = -0.0000002667$, is very, very accurate, almost beyond any current needs. Note the pattern of the number, which uses the digits 1, 1, 3, 3, 5, 5, stacked to make the fraction. Now, how good are these? Can we do better? Well, $3\frac{1}{6} = \frac{19}{6}$ is not nearly as accurate as $\frac{22}{7}$. Yet, $\frac{22}{7}$, is the most accurate fraction approximation up to $\frac{179}{57}$, but the improved accuracy of the latter fraction is slight. (Ask your students to compute these differences. They will begin to use very small numbers.) On the other hand, the next better approximation than $\frac{355}{113}$ is the whopping big fraction $\frac{53228}{16943}$, and as before the improvement is only slight. This should give the idea that these two revered fractions, $\frac{22}{7}$ and $\frac{355}{113}$, have a special place in the world of approximations.

## MODERN TIMES

The current, best approximation to $\pi$ is accurate to 1,240,000,000,000 places. (That's more than a trillion digits.) To give an idea how many digits these are, typing them all out at 10 digits to the inch, the entire approximation would run 1,957,070 miles. This is almost 78 times around the earth, or four round trips to the moon! The method uses a complex formula involving the arc tangent function. It was computed a HITACHI SR8000/MP supercomputer under the project direction of Yasumasa Kanada. See http://www.super-computing.org/ for more information.

## ACTIVITIES

1. Make the decimal approximations of all the fractions above. Compare the decimals to one another and then to $\pi$ itself.
2. Ask your students to recreate the clever diagram of the ancient Egyptians.

3. In the 6th century, Indian mathematicians used this description, "Add 4 to 100, multiply by 8, and add 62,000. The result is approximately the circumference of a circle of which the diameter is 20,000. What is their effective $\pi$? Answer: Computing, we have $\frac{(100+4)\times 8+62,000}{20,000}=3\frac{177}{1250}=3.1416$

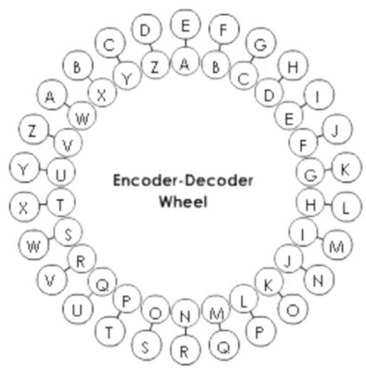
READINGS

"A History of Pi";
http://www-groups.dcs.stand.ac.uk/history/HistTopics/Pithroughtheages.html

G. DONALD ALLEN

# 3. THE CAESAR CYPHER

A **cypher** (or encryption) is a method of transforming a message into a set of alternate characters that conceals the contents of the message. The **Caesar cypher** (or **shift**) was one of the earliest cyphers ever used. More than two thousand years ago, Julius Caesar was able to convey secret messages to his generals and colleagues. It is simple and effective. Each character is shifted a specified number of places to the right, with the provision that at the end of the alphabet, the characters “wrap around” to the beginning of the alphabet. This is shown below for a shift of four places. So, “A” is shifted to “E”; “B” is shifted to “F”, “W” is shifted to “A”, and so on. For example, the message “Send more money” is encrypted to “Wirh qsvi qsric.”




wheel

Shift by four characters

An alternative to the Encoder-Decoder wheel is a linear representation of the shift.

| Original | A | B | C | D | E | F | G | H | I | J | K | L | M | N | O | P | Q | R | S | T | U | V | W | X | Y | Z |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| New | E | F | G | H | I | J | K | L | M | N | O | P | Q | R | S | T | U | V | W | X | Y | Z | A | B | C | D |

However, the cypher is now easy to decipher using frequency analysis. That is, the number of each letter occurrence is counted and compared with standard frequency counts for normal text. For example, “E” is the most commonly occurring letter, occurring 12.702% of the time. So, it is natural to guess that the letter in the encrypted message occurring most often is an “E.” This can help the cryptographer determine or guess the true letters. In the version of the Caesar cypher below, numbers are

*G. D. Allen & A. Ross (Eds.), Pedagogy and Content in Middle and High School Mathematics, 11–12.*

shifted as numbers, capital letters are shifted as capitals, and lower case letters are shifted as lower case. For each shift, the Encoder-Decoder wheel is shown. (Press "Show encoder wheel.") Frequency counts of letters for the English language can be found at http://en.wikipedia.org/wiki/Letter_frequencies

*Example.* With the four character shift, the message "I love American Idol," is encoded as "L oryh Dphulfdq Lgro."

*Where's the Math?* It lies in what is call **modular** arithmetic. Modular arithmetic is based on a specific modulus. We define

$$a = b \pmod{c}$$

to be the remainder of *b* divided by *c*. The **modulus** is **c**. For example 5 = 12 (mod 7) because 5 is the remainder of 12 divided by 7. For our present situation we are working with the twenty six letters of the alphabet; so our modulus *c* = 26. **Encode** the letters as the numbers 0, 1, 2, ... , 25 as shown in the chart below.

| Alphabet | a | b | c | d | e | f | g | h | i | j | k | l | m | n | o | p | q | r | s | t | u | v | w | x | y | z |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Encoding | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 | 20 | 21 | 22 | 23 | 24 | 25 |

So with a shift of four, each encoded letter is increased by four units. The letter "A" is encoded as "0" and this is shifted to "4" which is "E". Written in modular arithmetic this is 4 = 4 (mod 26). On the other hand, the letter "Y" is encoded as "24" and this is shifted by "4" which is 28. Now dividing by 26 gives the remainder 2 and this is decoded to "C". Written in modular arithmetic this is 2 = 28 (mod 26).

In summary, letters are **encoded** as numbers. To encrypt the letters, we perform modular arithmetic on the numbers, in this case add 4, and then the numbers are **decoded** back to letters. Essentially every modern encryption scheme uses the encoding of letters to numbers.

### TERMINOLOGY:

- Cypher (also spelled as *Cipher*) - Encryption
- Caesar cypher (cipher)
- Encode - Encoding
- Decode - Decoding
- Modular arithmetic
- Modulus

G. DONALD ALLEN

# 4. HOW BIG IS INFINITY?

Infinity is a relatively recent term in mathematics, having officially been around only since the nineteenth century. It was paradoxically created in an attempt to solve a complicated mathematical problem, but its presence was felt for centuries prior to that. It was felt in philosophy, religion, and in mathematics, partly due to the seeming need for infinite processes, infinite time span, infinitesimal divisibility of matter, and so on. In the months ahead, we can consider all of these if there becomes a need, but today we want to discover how big it is. Historically, infinity was a number, concept, or idea that could be approached but never reached or achieved. Today, it has attained its rightful place as a number, with precise rules on how to use it and what it means.

Everyone knows infinity is the biggest "thing" there is, no matter what the context. So, it matters not to discuss this aspect, the bigness. What is more fun is to consider some of the anomalies it creates.

## ARITHMETIC

To work numerically with infinity, mathematicians have created the *extended number system*, which consists of all real numbers plus infinity ($\infty$) and minus infinity ($-\infty$). For real numbers, the rules are the same as usual, but for the arithmetic involving $\infty$, we have for every real $a$

$$a + \infty = \infty$$

$$a * \infty = \pm\infty \text{ for } a \neq 0 \text{ with the } \pm \text{ generated by the sign of } a$$

$$\frac{a}{\infty} = 0$$

$$0 * \infty = 0$$

$$\infty + \infty = \infty$$

$$\infty - \infty = \text{ is undefined and } \frac{\infty}{\infty} \text{ is undefined}$$

With this set of rules, the extended real numbers form a consistent number system that obeys the laws of closure, commutativity, associativity, and distibutivity. The reason $\infty - \infty$ and $\frac{\infty}{\infty}$ are undefined is because we can't make any consistent sense

*G. D. Allen & A. Ross (Eds.), Pedagogy and Content in Middle and High School Mathematics, 13–15.*

of what it should be to maintain a consistent number system. For example, suppose we do what seems natural and define $\infty - \infty = 0$. Then, adding 1 to both sides of the equation gives $1 + \infty - \infty = 1 + 0$ . By associativity, we have $(1 + \infty) = \infty$. So, $1 + \infty - \infty = (1 + \infty) - \infty = \infty - \infty = 0$, but $1 + 0 = 1$. This implies $0 = 1$, which we know is not so. Selecting $\infty - \infty$ to be anything else similarly will result in a contradiction. Ditto for $\frac{\infty}{\infty}$.

## MAGNITUDE

We know $\infty$ is really, really big. But, it's bigger than that. One way of comparing the size of a basket of apples and a basket of pears is to put them in correspondence. In other words, you would place each apple next to a pear until one of the baskets is exhausted of fruit. Then, we can say that the basket with fruit remaining in it is the larger in size. Basically, we also do this by counting. However, there are records of some tribes of Indians, not knowing counting, who used this correspondence idea.

So, now let's suppose we have our baskets filled with an infinite quantity of objects. To make things easier, let's work with just the real numbers in the interval [0,1] and [0,2]. Clearly, both contain an infinite number of numbers, and just as clearly, the larger interval has "twice" the size. But, does it? We can't do the arithmetic $2\infty - \infty$ because this quantity is not defined. (Note. $2\infty - \infty = \infty - \infty$, and we've shown there can be no meaning to this.) Using the correspondence idea, we take any number $x$ in [0,1] and multiply it by 2 to get $2x$. Now, all the numbers in [0,1] have been put in correspondence with those in [0,2]. Not only is the correspondence perfect but, it is one-to-one. This remarkable observation shows that these two sets with one obviously "twice" the other have exactly the same number of points in them. (Note. Use rational numbers, and the argument is the same.)

*Definition*: An infinite number of numbers is called *countable* if it can be put into one-to-one correspondence with the natural numbers.

*Problem*: Show that there is exactly the same number of even positive integers as there are positive integers.

*Problem*: The famous Hotel Infinity, with an infinite number of rooms, is completely filled, and a new guest arrives requesting lodging for the night. The clerk says, "Of course, sir", and makes a room available. How did he do it?

## HIGHER ORDERS OF INFINITY

Infinity gets more bizarre when we consider all the real numbers in [0,1] and all the natural numbers, i.e., 1, 2, 3…. Both sets of numbers have an infinite number of members. This time, there is not a possible way to put them in a perfect correspondence. This means the infinity of [0,1] is *fundamentally larger* than that of the natural numbers. So there are magnitudes of infinity, just like there are magnitudes of numbers.

Let's show that the number of numbers in [0,1] cannot be put into one-to-one correspondence with the natural numbers. Each number in [0,1] has a decimal expansion. Assume we can put all of them in correspondence with the natural numbers. We express this assumption by writing $d_1, d_2, d_3, \ldots$ as all of the decimals written with their correspondent integer. Each decimal number has a full decimal expansion. So,

$$d_1 = 0.d_{11}d_{12}d_{13}\cdots$$
$$d_2 = 0.d_{21}d_{22}d_{23}\cdots$$

and so on, where all the digits $d_{ij}$ are integers 0, 1, 2, …, 9. Now, construct a brand-new decimal $f = f_1 f_2 f_3 \ldots$ by the rule that $f_1 \neq d_{11}$, $f_2 \neq d_{22}$, $f_3 \neq d_{33}$, and so on. In this way, $f$ cannot equal any of the $d_1, d_2, d_3, \ldots$, and this "proves" we cannot make the correspondence we assumed we could.

It was hardly 140 years ago that these ideas of orders of infinity turned mathematics on its axis. Infinity is now quite tamed; it is no longer the mystery it once was.

## REFERENCE

http://www.math.tamu.edu/~dallen/masters/infinity/content2.htm

GEORGE TINTERA

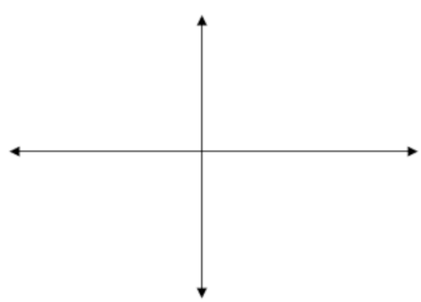
# 5. MAKING COMPLEX ARITHMETIC REAL!

With $i = \sqrt{-1}$ being called an imaginary number, it's easy to see why many students don't view complex numbers as actual numbers and/or take any interest in them. Such complex analysis is an important tool in engineering, and students need real-world examples that will motivate them to study the topic of complex numbers. We might humor our students by telling them that the real reason they need to know about complex numbers is to pass the next test or class. Instead, we should offer some simple exercises that give insight into the value of complex numbers.

We can associate with each complex number, $z = a + bi$, the point $(a, b)$, in the plane. The real number a is called the real part of z and the real number b is called the imaginary part of z. Adding two complex numbers $z = a + bi$ and $w = c + di$ is done by adding the real parts and imaginary parts: $z+w = (a+c) + (b+d)i$. Combining this definition of arithmetic and the geometry of the numbers themselves leads to the following activities.

Plot points associated with the following complex numbers: $z_1 = 1 + i$, $z_2 = –1 + i$, $z_3 = –1 – i$, and $z_4 = 1 – i$. Add the complex number $w = 2+3i$ to each of the numbers $z_1$ to $z_4$, plotting the sums as well. To complete the picture, connect the numbers $z_1$ to $z_4$ as vertices of a square. Connect the sums in the same way. The resulting picture gives us our first clue as to the power of complex arithmetic: adding a complex number translates another complex number or set of numbers. This is a way of doing geometry with arithmetic.

| z | z+(2+3i) |
|---|---|
| 1+i | |
| –1+i | |
| –1–i | |
| 1–i | |

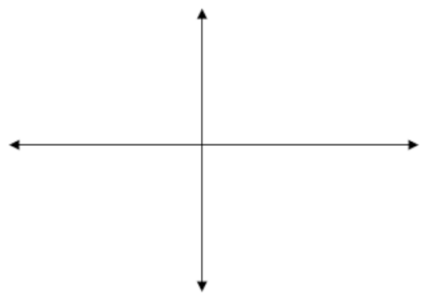


What about complex multiplication? Multiplying two complex numbers $z = a + bi$ and $w = c + di$ is performed the same way as multiplication of any two real binomials, while also using the identity, $i^2 = –1$. We can verify that $zw = (ac–bd) + (ad+bc)i$.

*G. D. Allen & A. Ross (Eds.), Pedagogy and Content in Middle and High School Mathematics, 17–19.*

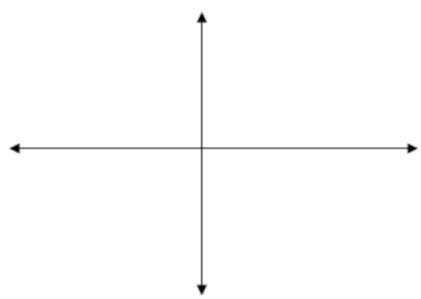
Start with a clean plot of the points $z_1$ to $z_4$ again. Multiply each of them by the complex number w = $\sqrt{2}/2+\sqrt{2}/2i$ Again, connect the numbers $z_1$ to $z_4$ as vertices of a square, and do the same for the resulting products. The resulting picture gives us more evidence of the power of complex arithmetic: multiplying by a complex number can rotate another complex number or set of numbers. We have extended the reach of arithmetic into geometry.

| z | z*w |
|---|---|
| 1+i | |
| –1+i | |
| –1–i | |
| 1–i | |



If we repeat the multiplication exercise with w = 1 + 1i = 1+i instead, we get a slightly different result. Instead of a simple rotation, the original square is stretched as well. Yes, there is a connection between the multiplication and, $\sqrt{2}\left(\sqrt{2}/2+\sqrt{2}/2i\right)=1+i$ and the stretch. This gives us evidence that multiplying by a real number dilates (stretches/contracts) another complex number or set of complex numbers.

| z | z*(1+i) |
|---|---|
| 1+i | |
| –1+i | |
| –1–i | |
| 1–i | |



Who knew that it would be possible to translate, rotate or stretch a figure just by arithmetic? And it doesn't end there. We might wonder if it is possible to do reflections, inversions or other geometric transformations with complex numbers. Why, of course.

Here are some exercises to work on. In them, S refers to the square in the plane, with vertices $z_1$ to $z_4$, as above.

Suppose there is a translation of S so that $z_1$ has image, –3 + 2i. What complex number, w, could be added to the vertices of S for that translation?