Mathematical Problems from Applied Logic I

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Mathematical Problems from Applied Logic I

Logics for the XXIst Century

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Mathematical Problems from Applied Logic I
New Logics for the XXIst Century

Two volumes of the *International Mathematical Series* present the most important thematic topics of logic confronting us in this century, including problems arising from successful applications areas such as Computer Science, AI language, etc. etc.

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WHAT IS ON MY MIND AS MOST
STRIKING/IMPORTANT/PRESSING
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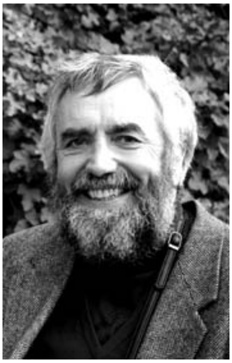

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[†] The endless cycle of death and rebirth to which life in the material world is bound. (OED)

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Nonstandard Inferences in Description Logics: The Story So Far

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Description logics (DLs) are a successful family of logic-based knowledge representation formalisms that can be used to represent the terminological knowledge of an application domain in a structured and formally well-founded way. DL systems provide their users with inference procedures that allow to reason about the represented knowledge. Standard inference problems (such as the subsumption and the instance problem) are now well-understood.

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Their computational properties (such as decidability and complexity) have been investigated in detail, and modern DL systems are equipped with highly optimized implementations of these inference procedures, which—in spite of their high worst-case complexity—perform quite well in practice.

In applications of DL systems it has turned out that building and maintaining large DL knowledge bases can be further facilitated by procedures for other, nonstandard inference problem, such as computing the least common subsumer and the most specific concept, and rewriting and matching of concepts. While the research concerning these nonstandard inferences is not as mature as the one for the standard inferences, it has now reached a point where it makes sense to motivate these inferences within a uniform application framework, give an overview of the results obtained so far, describe the remaining open problems, and give perspectives for future research in this direction.

1. Introduction

Description logics (DLs) [**12**] are a family of knowledge representation languages which can be used to represent the terminological knowledge of an application domain in a structured and formally well-understood way. The name *description logics* is motivated by the fact that, on the one hand, the important notions of the domain are described by *concept descriptions*, i.e., expressions that are built from atomic concepts (unary predicates) and atomic roles (binary predicates) using the concept and role constructors provided by the particular DL. For example, the concept of "a man that is married to a doctor, and has only happy children" can be expressed using the concept description

$$\mathsf{Man} \sqcap \exists \mathsf{married}.\mathsf{Doctor} \sqcap \forall \mathsf{child}.\mathsf{Happy}.$$

On the other hand, DLs differ from their predecessors, such as semantic networks and frames [**84, 79**], in that they are equipped with a formal, *logic*-based semantics, which can, for example, be given by a translation into first-order predicate logic. For example, the above concept description can be translated into the following first-order formula (with one free variable x):

$$\begin{aligned}\mathsf{Man}(x) \wedge \exists y.(\mathsf{married}(x,y) \wedge \mathsf{Doctor}(y))\\ \wedge \forall y.(\mathsf{child}(x,y) \rightarrow \mathsf{Happy}(y)).\end{aligned}$$

In addition to the formalism for describing concepts, DLs usually also provide their users with means for describing individuals by stating to which concepts they belong and in which role relationships they participate. For example, the assertions

$$\mathsf{Man}(\mathsf{JOHN}), \quad \mathsf{child}(\mathsf{JOHN}, \mathsf{MARY}), \quad \mathsf{Happy}(\mathsf{MARY})$$

state that the individual John has a child Mary, who is happy.

Knowledge representation systems based on description logics (DL systems or DL reasoners) [**95, 81**] provide their users with various inference capabilities that deduce implicit knowledge from the explicitly represented knowledge. Standard inference services are *subsumption* and *instance checking*. Subsumption allows the user to determine subconcept-superconcept relationships, and hence, compute a subconcept-superconcept hierarchy: C is subsumed by D if and only if all instances of C are also instances of D, i.e., the first description is always interpreted as a subset of the second description. Instance checking asks whether a given individual necessarily belongs to a given concept, i.e., whether this instance relationship logically follows from the descriptions of the concept and of the individual.

In order to ensure a reasonable and predictable behavior of a DL reasoner, these inference problems should at least be decidable for the DL employed by the reasoner, and preferably of low complexity. Consequently, the expressive power of the DL in question must be restricted in an appropriate way. If the imposed restrictions are too severe, however, then the important notions of the application domain can no longer be expressed. Investigating this trade-off between the expressivity of DLs and the complexity of their inference problems has been one of the most important issues of DL research in the 1990s. As a consequence of this research, the complexity of reasoning in various DLs of different expressive power is now well-investigated (see [**49**] for an overview of these complexity results). In addition, there are highly optimized implementations of reasoners for very expressive DLs [**61, 54, 62**], which—despite their high worst-case complexity—behave very well in practice [**60, 53**].

DLs have been applied in many domains, such as medical informatics, software engineering, configuration of technical systems, natural language processing, databases, and web-based information systems (see Part III of [**12**] for details on these and other applications). A recent success story is the use of DLs as ontology languages [**15, 16**] for the Semantic Web [**33**]. In particular, the W3C recommended ontology web language OWL [**64**] is based on an expressive description logic [**67, 66**].

Editors—such as OilEd [**32**] and the OWL plug-in of Protègè [**69**]—supporting the design of ontologies in various application domains usually allow their users to access a DL reasoner, which realizes the aforementioned *standard inferences* such as subsumption and instance checking. Reasoning is not only useful when working with "finished" ontologies, it can also support the ontology engineer while building an ontology, by pointing out inconsistencies and unwanted consequences. The ontology engineer can thus use reasoning to check whether the definition of a concept or the description of an individual makes sense.

However, these standard DL inferences—subsumption and instance checking—provide only little support for actually coming up with a first version of the definition of a concept. The non-standard inferences considered in this paper were introduced to overcome this deficit, by allowing the user to construct new knowledge from the existing one. Our own motivation for investigating these novel inferences comes from an application in chemical process engineering where a knowledge base has been built by different knowledge engineers over a rather long period of time [**87, 71, 80, 44, 35, 77, 94**].

The goal of this paper is

(i) to motivate nonstandard inferences by means of a simple application scenario,
(ii) to provide an overview of the results that have been obtained for nonstandard inferences so far, and
(iii) to explain the main techniques employed for solving these novel inference problems.

In order to be able to describe the latter in detail, the exposition of the techniques is mainly restricted to the DL $\mathcal{ALE}$. However, we also provide references to results for other DLs.

Structure of the paper

In Section 2, we introduce typical DL constructors and the most important standard inference problems. In addition, we give a brief review of the different approaches for solving these inference problems, and of their complexity in different DLs. In Section 3, we first motivate the need for nonstandard inferences in a typical application scenario, and then formally define the most important nonstandard inferences in description logics. Then, we briefly introduces the techniques used to solve these problems. Since these techniques depend on a syntactic characterization of the subsumption problem, Section 3 is followed by a section that describes such a characterization for the DL $\mathcal{ALE}$, which we use as a prototypical example (Section 4). The next four sections consider the four most important nonstandard inference problems: computing the *least common subsumer* and the *most specific concept*, *rewriting*, and *matching*. Related nonstandard inferences are briefly discussed in the respective sections as well. We explain the results on these four nonstandard inferences in $\mathcal{ALE}$ in detail, whereas results for other DLs are reviewed only briefly. Finally, Section 9 summarizes the results on nonstandard inferences obtained so far, and gives perspectives for further research.

2. Description Logics and Standard Inferences

In order to define concepts in a DL knowledge base, one starts with a set N_C of concept names (unary predicates) and a set N_R of role names (binary predicates), and defines more complex *concept descriptions* using the *concept constructors* provided by the concept description language of the particular system. In this paper, we consider the DL $\mathcal{ALCN}$ and some of its sublanguages. *Concept descriptions* of $\mathcal{ALCN}$ are built using the constructors shown in the first part of Table 1. In this table, r stands for a role name, n for a nonnegative integer, A for a concept name, and C, D for arbitrary concept descriptions.

A *concept definition* $A \equiv C$ (as shown in the second part of Table 1) assigns a concept name A to a complex description C. A finite set of such definitions is called a *TBox* if and only if it is unambiguous, i.e., each name has at most one definition. The concept names occurring on the left-hand side of a concept definition are called *defined* concepts, and the others *primitive*. In many cases, one restricts the attention to *acyclic* TBoxes, where the definition of a defined concept A cannot (directly or indirectly) refer to A itself.

A (concept or role) *assertion* is of the form shown in the last part of Table 1. Here, a, b belong to an additional set N_I of individual names. A finite set of such assertions is called an *ABox*.

The *sublanguages* of $\mathcal{ALCN}$ that will be considered in this paper are shown in Table 2. The first column explains the naming scheme for the members of the $\mathcal{AL}$-family.

The *semantics* of concept descriptions is defined in terms of an *interpretation* $\mathcal{I} = (\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}})$. The domain $\Delta^{\mathcal{I}}$ of $\mathcal{I}$ is a non-empty set and the interpretation function $\cdot^{\mathcal{I}}$ maps each concept name $A \in N_C$ to a set $A^{\mathcal{I}} \subseteq \Delta^{\mathcal{I}}$, each role name $r \in N_R$ to a binary relation $r^{\mathcal{I}} \subseteq \Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}}$, and each individual name $a \in N_I$ to an element $a^{\mathcal{I}} \in \Delta^{\mathcal{I}}$. The extension of $\cdot^{\mathcal{I}}$ to arbitrary concept descriptions is inductively defined, as shown in the third column

Name	Syntax	Semantics
top-concept	$\top$	$\Delta^{\mathcal{I}}$
bottom-concept	$\bot$	$\emptyset$
negation	$\neg C$	$\Delta^{\mathcal{I}} \setminus C^{\mathcal{I}}$
atomic negation	$\neg A$	$\Delta^{\mathcal{I}} \setminus A^{\mathcal{I}}$
conjunction	$C \sqcap D$	$C^{\mathcal{I}} \cap D^{\mathcal{I}}$
disjunction	$C \sqcup D$	$C^{\mathcal{I}} \cup D^{\mathcal{I}}$
value restriction	$\forall r.C$	$\{x \in \Delta^{\mathcal{I}} \mid \forall y : (x,y) \in r^{\mathcal{I}} \to y \in C^{\mathcal{I}}\}$
existential restriction	$\exists r.C$	$\{x \in \Delta^{\mathcal{I}} \mid \exists y : (x,y) \in r^{\mathcal{I}} \wedge y \in C^{\mathcal{I}}\}$
at-least restriction	$\geqslant n\, r$	$\{x \in \Delta^{\mathcal{I}} \mid \sharp\{y \mid (x,y) \in r^{\mathcal{I}}\} \geqslant n\}$
at-most restriction	$\leqslant n\, r$	$\{x \in \Delta^{\mathcal{I}} \mid \sharp\{y \mid (x,y) \in r^{\mathcal{I}}\} \leqslant n\}$
concept definition	$A \equiv C$	$A^{\mathcal{I}} = C^{\mathcal{I}}$
concept assertion	$C(a)$	$a^{\mathcal{I}} \in C^{\mathcal{I}}$
role assertion	$r(a,b))$	$(a^{\mathcal{I}}, b^{\mathcal{I}}) \in r^{\mathcal{I}}$

TABLE 1. Syntax and semantics of concept descriptions, definitions, and assertions

of Table 1. In the rows treating at-least and at-most number restrictions, $\sharp M$ denotes the cardinality of a set M.

The interpretation $\mathcal{I}$ is a *model* of the TBox $\mathcal{T}$ if it satisfies all its concept definitions, i.e., $A^{\mathcal{I}} = C^{\mathcal{I}}$ for all $A \equiv C$ in $\mathcal{T}$, and it is a *model* of the ABox $\mathcal{A}$ if it satisfies all its assertions, i.e., $a^{\mathcal{I}} \in C^{\mathcal{I}}$ for all concept assertions $C(a)$ in $\mathcal{A}$ and $(a^{\mathcal{I}}, b^{\mathcal{I}}) \in r^{\mathcal{I}}$ for all role assertions $r(a,b)$ in $\mathcal{A}$.

Based on this semantics, we can now formally introduce the standard inference problems in description logics.

Definition 2.1. Let $\mathcal{A}$ be an ABox, $\mathcal{T}$ a TBox, C, D concept descriptions, and a an individual name.

- C is *satisfiable* w.r.t. $\mathcal{T}$ if there is a model $\mathcal{I}$ of $\mathcal{T}$ such that $C^{\mathcal{I}} \neq \emptyset$.
- D *subsumes* C w.r.t. $\mathcal{T}$ ($C \sqsubseteq_{\mathcal{T}} D$) if $C^{\mathcal{I}} \subseteq D^{\mathcal{I}}$ for all models $\mathcal{I}$ of $\mathcal{T}$.