SVEN BODO WIRSING

MAXIMAL NILPOTENT SUBALGEBRAS

A correspondence theorem within solvable associative algebras. With 242 exercises

II

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Thank you Dad

Thank you Dad,
For everything you've done,
Thank you Dad,
From your thankful son.

Thank you Dad,
For just being there,
Thank you Dad,
For showing that you care.

Thank you Dad,
For showing us the way,
Thank you Dad,
For working hard every day.

Thank you Dad,
You're the very best,
Thank you Dad,
Because you're none like the rest.

Contents

Introduction

Within series I we have focussed on the following two main topics: the determination of the Cartan subalgebras and of the nilradical of the associated Lie algebra A° based on a finite-dimensional associative unitary algebra A. Both Lie substructures are maximal nilpotent in A°: Cartan subalgebras with respect to the subalgebra lattice and the nilradical with respect to the ideal lattice of A°. If the factor algebra by the nilradical of A is separable, then – by using the theorem of Wedderburn-Malcev – a radical complement T of $rad(A)$ in A exists. Based on this radical complement we were able to determine within series I the Cartan subalgebras and the nilradical of A° for several classes of algebras A. In particular, if A is solvable (which is the case of $A/rad(A)$ and T being commutative) we have proven that the centralizers of the radical complements – denoted by $C_A(T)$ – are exactly the Cartan subalgebras of A°. This results was proven originally by Thorsten Bauer within his dissertation [4]. In particular, all Cartan subalgebras of A° are associative subalgebras of A. The theorem of Wedderburn-Malcev is used to prove further that all Cartan subalgebras of A° are conjugated under the group $1 + rad(A)$. If we focus on the central part of T in A – which is $Z(A) \cap T$ – we have derived in series I additionally that this part is separable and that the nilradical of A° is the inner direct sum of $rad(A)$ and $Z(A) \cap T$. Cartan subalgebras are maximal Lie nilpotent subalgebras. If A is solvable, then the nilradical of A° is a maximal Lie nilpotent subalgebra, too.

Within this series we will enhance this theory of maximal nilpotent subalgebras of A° in the **solvable case** of A further. The following questions are the guidelines of this series related to the associated Lie algebra A° and also to the group of units $E(A)$ of A:

· In what way can we determine **all** maximal nilpotent Lie subalgebras of A°?

· Does a special or extremal position of the nilradical and the Cartan subalgebras exist among all maximal nilpotent Lie subalgebras of A°?

· In what way can we determine the Carter subgroups and the Fitting subgroup of $E(A)$? Is the Fitting subgroup a maximal nilpotent subgroup?

· In what way can we determine **all** maximal nilpotent subgroups of $E(A)$?

· Does a special or extremal position of the Fitting subgroup and the Carter subgroups exist among all maximal nilpotent subgroups of $E(A)$?

· Do structural connections exist between maximal nilpotent subalgebras of A° and maximal nilpotent subgroups of $E(A)$?

The intention of chapter 1 is to summarize special structures like group algebra, the Solomon algebra or the Solomon Tits algebra. These algebras are used to visualize the results within this work and to guide the reader within the exercises to a deeper insight of the proven results.

For the analysis of structural connections between maximal nilpotent subgroups and Lie subalgebras we will use the main result of chapter 2 frequently in this work: the theorem of Xiankun Du proven in 1992 based on radical algebras comprised that the upper central chain of the associated Lie algebra coincide with the upper central chain of the quasi regular group – or also called star or circle group (which is a generalization of the group of units) – in every step. In particular, the class of nilpotency of both structures is identical. This result was conjectured by Stephen Arthur Jennings 40 years ago and partly proven by Hartmut Laue in the eighties. Oftentimes, it is simpler to do calculations in the Lie algebra and not within the circle group. For example, radicals of associative algebras are radical algebras. In the context of maximal nilpotent substructures we use the result to derive a connection between the nilpotency classes of maximal nilpotent Lie subalgebras and maximal nilpotent subgroups. As an excursus at the end of chapter 2 we derive another application of the theorem of Xiankun Du. If we focus on the upper central chain of the circle group of a radical algebra and here on the factor groups of the $(n+1)$-th modulo the n-th center, then these factor groups are – by definition of the upper central chain – abelian groups. In the case of a radical algebra based on a field of positive characteristic p we can derive – by using the theorem of Xiankun Du – that these factor groups are indeed of exponent p. Applied to the group algebra – for which Adalbert Bovdi has published this result – the reader may prove this result within the exercises and experience the transfer of group theoretic questions to Lie theory.

As aforementioned, the guidelines of this work are connected to solvable associative algebras. The main focus will be to analyze structural properties of and connections between the associative and the associated Lie as well as the derived group structure in form of the group of units concerning maximal nilpotency. For the solvability itself a connection between these three structures is existing: we will prove within chapter 3 that the solvability for the associative algebra, its associated Lie algebra and its group of units based on a finite-dimensional associative unitary K-algebra (for a field K possessing at least 5 elements and $char(K) \neq 2$) are equivalent. This result was one incentive for our guidelines. As an excursus we focus at the end of chapter 3 on a connection between maximal solvable Lie subalgebras and maximal solvable subgroups: the so-called Borel subalgebras of A° – which are maximal solvable Lie subalgebras – are indeed associative unital subalgebras of A based for fields of characteristic zero. For proving this, we need a theorem of Sophus Lie and a result of Hartmut Laue concerning the associative algebra span. The group of units of the Borel subalgebras are solvable groups. Unfortunately, the proof that they are maximal solvable subgroups – which are so-called Borel subgroups – was not possible to perform. But we could prove that each Borel subalgebra leads to a new group of units. The reason is that the K-space generated by the group of units is the whole algebra. This approach – creating the group of units and the K-space generated by them – will often be useful within this work for describing and analyzing the connections between subalgebras and subgroups.

Thorsten Bauer has already analyzed one guideline of this work within his dissertation [4]: the determination of the Carter subgroups of the group of units of an unital finite-dimensional associative solvable algebra possessing a separable factor algebra by its nilradical. He has proven that the Carter subgroups – for a field possessing at least three elements – are exactly the group of units of the Cartan subalgebras of the associated Lie algebra. The assumption for the field is necessary to ensure that the algebra is generated by its group of units. Thus, the result of Thorsten Bauer can be reformulated as follows: the K-space generated by the Carter subgroups are exactly the Cartan subalgebras. Again, the concept of **creating the group of units and creating the K-space generated by the group of units** arise. Within the article [5] of Thorsten Bauer and Salvatore Siciliano concerning the determination of the Carter subgroups a result is proven which will be of significant importance later on in this work, too: the K-space generated of a nilpotent subgroup based on a finite-dimensional associative solvable K-algebra is Lie nilpotent.

The phenomenon of connecting Cartan subalgebras and Carter subgroups arise for the nilradical and the Fitting subgroup, too. We will prove within chapter 5 that both structures are connected via creating the group of units and the creating the K-space based on the group of units. The result of Thorsten Bauer and Salvatore Siciliano concerning the K-space generated by a nilpotent subgroup will be of significant importance for proving this connection.

The previous chapters have focussed on special and prominent examples of maximal nilpotent substructures within the group of units and the associated Lie algebra. In this chapter we analyze more generally the construction, determination and characterization of all maximal nilpotent Lie subalgebras. In a first step we prove – in analogy to the Borel subalgebras stated earlier (but based on a completely different argumentation) – that maximal Lie nilpotent subalgebras are unital associative subalgebras. Thus, we are able to use results of series I concerning these special associative subalgebras: the inner structure of these associative subalgebras M of A is presentable as the inner direct sum of its nilradical $rad(M)$ (which is contained in $rad(A)$ by using the solvability of A) and the unique and central radical complement $VSEP(M)$ consisting of fully separable elements: $M = rad(M) \oplus VSEP(M)$. The theorem of Wedderburn-Malcev lets us derive that $VSEP(M)$ is contained in a radical complement T of $rad(A)$ in A. Based on the inner structure of M and the radical complement T we can prove that a Lie nilpotent associative subalgebra M is maximal Lie nilpotent if and only if the centralizer conditions $C_{rad(A)}(VSEP(M)) = rad(M)$ and $C_T(rad(M)) = VSEP(M)$ are valid. A simple but remarkable consequence is that maximal nilpotent Lie subalgebras satisfy the double-centralizer conditions $C_{rad(A)}(C_T(rad(M))) = rad(M)$ and $C_T(C_{rad(A)}(VSEP(M))) = VSEP(M)$. For determining all maximal Lie nilpotent subalgebras we use these centralizer and double-centralizer properties: we start with an unital subalgebra C of T and calculate the double-centralizer $C_T(C_{rad(A)}(C))$. We proceed by calculating the double-centralizer of the double-centralizer again and again. This process must be stable of finite many steps because of the finite dimension of A. If the process is stable, then the resulting subalgebra $\hat{C}$ in T combined with the direct summand $C_{rad(A)}(\hat{C})$ is maximal Lie nilpotent. The dual process – beginning with a subalgebra of $rad(A)$ – leads also to maximal Lie nilpotent subalgebras, but not to new ones. A natural question is to determine the number of steps after which the double-centralizing is stable. The answer is simple: not from the beginning but after the first double-centralizing. Thus, we have to use the double-centralizing on the lattice of unital subalgebras of T resp. the lattice of subalgebras of $rad(A)$ once and construct as already described all maximal Lie nilpotent subalgebras. The nilradical and the Cartan subalgebras have an extremal position among all maximal Lie nilpotent subalgebras. The component of the nilradical resp. Cartan subalgebras in T is central in A resp. the whole radical complement. Within the nilradical its extremely large resp. small (and therefore dual). For all other maximal nilpotent subalgebras the part in T resp. $rad(A)$ is situated between these two values. By using another radical complement only isomorphic copies of maximal nilpotent Lie subalgebras arise (based on the theorem of Wedderburn-Malcev). Hence, all isomorphic classes of maximal nilpotent subalgebras can be bounded by the number of unital subalgebras of a fixed radical complement T. This number is finite because T is separable and commutative: T is a so-called futile algebra. We prove this statement within a separate section and estimate this number by the upper bound $B(dim_K(T))$ – which are the so-called Bell numbers.

In chapter 7 we present a bijective connection between maximal nilpotent Lie subalgebras and maximal nilpotent subgroups. It becomes apparent that – as already stated for the Cartan subalgebras and the Carter subgroups resp. the nilradical and the Fitting subgroup – there is a general connection between maximal nilpotent substructures: the group of units of maximal Lie nilpotent subalgebra (which is indeed an unital associative subalgebra) is a maximal nilpotent subgroup and the K-space generated by a maximal nilpotent subgroup is a maximal nilpotent Lie subalgebra (Here we will use the already mentioned result of T. Bauer and S. Siciliano again.). In addition, this connection is bijective: the functions $E(\cdot)$ – creating the group of units – and $\langle \cdot \rangle_K$ – creating the K-space generated by the group of units – are inverse to each other. By using the theorem of Xiankun Du we derive the more deeper insight that the classes of nilpotency of two connected maximal nilpotent substructures are identical. The results presented in chapter 6 can be transferred by using this connection to maximal nilpotent subgroups which is the content of chapter 8.

Thus, in analogy to chapter 6 we describe within chapter 8:

· the inner structure of the maximal nilpotent subgroups as direct products of unipotent and central, fully separable elements,

· the characterization of maximal nilpotent subgroups by manifold centralizers,

· the determination of all maximal nilpotent subgroups by double-centralizing all subgroups of $E(T)$ and combining the centralized unipotent part to it,

· the dual determination of all maximal nilpotent subgroups by double-centralizing all subgroups of $1 + rad(A)$ and combining the centralized fully-separable part to it,

· the extremal position of the Carter subgroups and the Fitting subgroup among all maximal nilpotent subgroups of $E(A)$,

· the behavior of the maximal nilpotent subgroups by changing the radical complement and

· the finiteness of the number of isomorphic classes of maximal nilpotent subgroups which can be bonded by Bell numbers.

The last chapter is dedicated to other prominent maximal nilpotent subgroups which are the so-called nilpotent injectors, nilpotent projectors and the Fischer subgroups. We will prove that they coincide with the Carter subgroups resp. the Fitting subgroup. Afterwards these prominent maximal nilpotent substructures are also defined for Lie algebras (nilpotent Lie injectors, nilpotent Lie projectors and Fischer subalgebras), and we will prove that they coincide with the Cartan subalgebras and the Lie nilradical. Posthumous, we derive the result that the group of units of the Fischer subalgebras, the nilpotent Lie projectors and the nilpotent Lie injectors are exactly the Fischer subgroups, the nilpotent projectors and injectors. Vice versa, the K-space generated by them is exactly the Fischer subalgebras, the nilpotent Lie projectors and the nilpotent Lie injectors.

As stated earlier we illustrate our results by using standard examples. These are mainly group algebras, the algebras of upper and lower triangular matrices over a field, the Solomon algebra in characteristic zero and the Solomon-Tits algebra. Within the first chapters these examples are investigated on a high detailed level, but within the last four chapters we use them only exemplary. A detailed analysis needs a deeper insight, and the author decided not to disconnect the reader from the general theory too far, but to do this analysis in series III.

Some applications are also transferred to the exercises at the end of each section or chapter. There are some exercises included enhancing the theory presented so far such that the reader can experience a deeper insight. In addition, at the beginning of each exercise series some open-ended topics are included which can be used by the reader – and also by the author – to do additional researches within this theory. The author has included some manually created graphics – mostly so called Hasse diagrams – to visualize the main results of this work.

Excercise 1 *What are the answers for the guidelines of this work?*

Chapter 1

Standard examples, symbols and notations

This chapter has a preliminary function by summarizing those monoids, groups, associative and Lie algebras which will arise frequently in this work. They will be used as examples for the proven theorems as well as for the exercises in which the reader shall apply the general results to them. In addition, we list the symbols and notations used in this series.

Sets and numbers

Let A, B, T be sets and $i, n, k \in \mathbb{N}_0$. We use the following symbols linked to set and number theoretical topics:

· $\emptyset$ - the empty set

· $A \cap B$ - intersection of A, B

· $A \cup B$ - union of A, B

· $A \setminus B$ - difference of A, B

· $A \times B$ - cartesian product of A, B

· $P(A)$ - power set of A

· $\underline{n}$ - the first n natural numbers

· $\underline{n}_0$ - the first n natural numbers including zero

· $p(n)$ - the number of partitions of n

· $n!$ - factorial of n

· $\mid A \mid$ - number of elements of A

· $S(n,k)$ - the kth-Stirling number of n

· $B(n)$ - the nth-Bell number

· $\binom{n}{k}$ - n choose k

· $\binom{T}{i}$ - the set of subsets of order i of T

· $\equiv$ - equivalent

· $\mod$ - modulo.

Groups and monoids

Let $p \in \mathbb{P}$, $n \in \mathbb{N}$, N be a set, M a monoid, G a group, N a normal subgroup of G, $a, b \in G$, $U, V \leq G$, A an associative unitary K-algebra, $c \in K$ and q a prime power number. The following monoids, groups and symbols are used:

· $st(G)$ - solvable class of G

· $cl(G)$ - nilpotency class of G

· $(Z_n(G))_{n \in \mathbb{N}}$ - ascending central chain of G

· $(G^{(n)})_{n \in \mathbb{N}}$ - descending central chain of G

· $(\gamma_n(G))_{n \in \mathbb{N}}$, $(G^{[n]})_{n \in \mathbb{N}}$ - commutator or derived series of G

· $\star$, $\star_c$ - star or circle composition

· $[a,b]$ - commutator of a, b

· $[U,V]$ - commutator of U, V

· a^{-1} - inverse element of a

· a^b - conjugate element of a with b

· $1 + rad(A)$ - normalized units

· $C_U(V)$ - centralizer of V in U

· $N_U(V)$ - normalizer of V in U

· $\mathcal{G}$ - class of groups

· $G^{'}$ - derived subgroup of G

· $(F^n(G))_{n \in \mathbb{N}}$ -Fitting series of G