Analyzing Multidimensional Well-Being

A Quantitative Approach

Satya R. Chakravarty

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This edition first published 2018

Registered Offices
John Wiley & Sons, Inc., 111 River Street, Hoboken, NJ 07030, USA

Editorial Office
111 River Street, Hoboken, NJ 07030, USA

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Library of Congress Cataloging-in-Publication Data

Names: Chakravarty, Satya R., author.
Title: Analyzing multidimensional well-being : a quantitative approach / by Satya R. Chakravarty.
Description: Hoboken, NJ : John Wiley & Sons, Inc., 2018. | Includes bibliographical references and index.
Identifiers: LCCN 2017033909 (print) | LCCN 2017019107 (ebook) | ISBN 9781119256908 (cloth) | ISBN 9781119256946 (pdf) | ISBN 9781119256953 (epub)
Subjects: LCSH: Social indicators. | Economic indicators. | Quality of life–Statistical methods. | Well-being–Statistical methods.
Classification: LCC HN25.C449 2017 (ebook) | LCC HN25 (print) | DDC 306–dc23 LC record available at https://lccn.loc.gov/2017033909

Cover design by Wiley
Cover image: © Wasant/Shutterstock

Set in 10/12pt WarnockPro by SPi Global, Chennai, India

Printed in the United States of America

10 9 8 7 6 5 4 3 2 1

In loving memory of my eldest brother Dr. Keshab R. Chakravarty, whose honesty and dedication towards profession have always been an inspiration to me.

Contents

Preface

Since often an income distribution of a society fails to include heterogeneity in the distributions of one or more other dimensions of well-being of a population such as health, education, and housing etc.; income's unsuitability as the solitary attribute of well-being is clearly understandable. In fact, it has now become well-recognized that human prosperity should be treated as a multidimensional aspect. Consequently, there has been a spur among researchers to work on multidimensional economic well-being.

There has been significant development in the areas of multivariate welfare, inequality, and poverty in the recent past, and hence, I felt the need to take the opportunity to delve deeper into the core values of the stated concepts. What is presented in this book is a theory of multidimensional welfare, inequality, and poverty in an axiomatic architecture. The aim is to clarify how we can proceed to the evaluation of the three issues and address the questions of enhancing welfare and reducing inequality and poverty.

The monograph casts ample light on the concepts, and I believe such an elusive discussion will intrigue students, teachers, researchers, and practitioners in the area. Substantive coverage of ongoing and advanced topics and their inquisitive, eloquent, accurate bestowal make the treatise theoretically and methodologically quite concurrent and comprehensive and highly susceptible to the practical problems of recent concern.

Since the use of simple one-dimensional indices for reckoning welfare and inequality of a population or the comparison of similar measurements pertaining to another population is an improper analysis, Chapter 1 looks to assess how well off a society can be in terms of individual achievements in different dimensions. The purpose of Chapter 2 is to review the alternative approaches to the evaluation of multidimensional inequality. Picking up from the note that welfare of a population needs an appraisal from a multivariate perspective, poverty can be regarded as a demonstration of inadequacy of achievements in different dimensions of wellbeing. Hence, Chapters 3 attempts to present an analytical discussion on the axiomatic approach to the measurement of multidimensional poverty.

There might arise a problem of gathering sufficient information on achievements in different dimensions of well-being, thereby raising questions at the poverty status of an individual. To tackle this ambiguity, fuzzy set theory can be employed to handle the vagueness resulting from obscurity. This fuzzy set approach to multidimensional poverty judgment has been addressed in Chapter 4.

In the persuasive investigations made in Chapters 3 and 4, which are based on the individual multidimensional achievements as inputs in a single period, time span of poverty is ignored. Nevertheless, there are plentiful reasons to believe that poverty is not a timeless concept. It can be regarded as a concept that endures changes over time. Furthermore, it would be wrong to expect that the transformations of income and nonincome dimensions of life will be the same across time. In view of this, Chapter 5 throws light on the different approaches that scrutinize lifetime poverty in a multidimensional framework.

Vulnerability and security risks always go hand in hand. In a wide sense, we can define vulnerability in terms of a system's disclosure and capability to cope sufficiently with discomfort. The study of vulnerability is hence quite significant because of the highly important follow-ups that may be generated as its implications for economic efficiency and long-term individual welfare. The purpose of Chapter 6 is deliberation of vulnerability as a multidimensional issue.

Finally, Chapter 7 reflects on the practicality of some composite and individualistic indices. A composite index is a summary measure, giving an all-inclusive picture of dimensional indices, associated with a dashboard. In the individualistic approach, individualwise indicators are derived initially by combining respective dimensional achievements and then by amalgamating the individual-level indicators.

At Paris School of Economics, I have had the excellent opportunity of working together with François Bourguignon, and I learned a great deal from talking with him. His influence has come not only through extensive discussions during the period but also through the use I made of the analytical framework in my later works that we developed. It is difficult for me to express my gratitude to him in words.

I have also worked jointly in this expanding area with Sabina Alkire, Mauricio Apablaza, Walter Bossert, Lidia Ceriani, Nachiketa Chattopadhyay, Conchita D'Ambrosio, Joseph Deutsch, Maria Ana Lugo, Diganta Mukherjee, Zoya Nissanov, Liu Qingbin, Ravi Ranade, Jacques Silber, Guanghua Wan, Gaston Yalonetzky, and Claudio Zoli. I have been very fortunate in having them as coauthors with whom I have had very illuminating conversations. I must acknowledge my extensive debt to them.

During the years, I have been privileged to receive comments and suggestions from Rolf Aaberge, Matthew D. Adler, Tony Atkinson, Valérie Berenger, Charles Blackorby, Kristof Bosmans, Florent Bresson, Koen Decancq, Stefan Dercon, David Donaldson, Jean-Yves Duclos, Indranil Dutta, Marc Fleurbaey,

James E. Foster, Mridu P. Goswami, Carlos Gradin, Nanak C. Kakwani, Serge-Christophe Kolm, Casilda Lasso de la Vega, François Maniquet, Ajit Mishra, Patrick Moyes, Conan Mukherjee, Erwin Ooghe, Rupayan Pal, Eugenio Peluso, Iñaki Permanyer, Krishna Pendakur, Martin Ravallion, Ernesto Savaglio, Amartya K. Sen, John A. Weymark, and Buhong Zheng. I would like to acknowledge the benefit I received from the interactions with them.

I benefitted a lot from the critiques of my students at the Bocconi University, Milan, Italy; Indian Statistical Institute, Kolkata, India; and Indira Gandhi Institute of Development Research, Mumbai, India. I am grateful to them for the joys and benefits that I derived from interactive teaching. The figure files were generated by Nandish Chattopadhyay and Snigdha Chatterjee sat through some sessions of proof corrections. It is a pleasure for me to acknowledge the help I received from them.

I must note the help and advice I have received from my wife Sumita and son Ananyo, whose influence is reflected throughout the book.

The book has been dedicated to the memory of my eldest brother Keshab R. Chakravarty, a renowned cardiologist, who has started his after-life journey on 1 August 2016 at 7.43 pm.

Kolkata *Satya R. Chakravarty*

Endorsements

1) "Analyzing Multidimensional Well-Being" by Satya Chakravarty provides a comprehensive review of a burgeoning new area of welfare economics and elaborates further on the way key unidimensional welfare concepts can be extended to the multidimensional case. An indispensable reference for all researchers interested in the measurement of social welfare and who feel the monetary focus is unduly restrictive.
   François Bourguignon, Emeritus Professor at Paris School of Economics, Former Chief Economist of the World Bank.
2) It has become the norm in the profession to define and measure well-being as a multidimensional concept instead of relying on income only. In his monograph, Satya Chakravarty provides us with a detailed, insightful, and pedagogical presentation of the theoretical grounds of multidimensional well-being, inequality, and poverty measurement. Any student, researcher, and practitioner interested in the multidimensional approach should begin their journey into such a fascinating theme with this wonderful book.
   François Maniquet, Professor, Catholic University of Louvain, Belgium.
3) This book starts from the premise that income cannot be the only indicator on which the measurement of well-being should be based. Other dimensions of well-being need to be taken into account, such as health, education, and housing. The implication of such a "Weltanschauung" is that well-being is a multidimensional phenomenon. But how should we then measure it? Satya Chakravarty, who has made fundamental contributions to this domain, gives us here a systematic presentation of the issues related to the measurement of multidimensional inequality, multidimensional poverty (with separate chapters on the fuzzy approach to poverty and poverty and time), multidimensional vulnerability, and composite indices such as the Human Development Index. In each chapter, the axioms underlying the various indices are clearly explained and the indices derived are well interpreted. In short, this is a remarkably rigorous and enlightening book

that should be required reading for anyone, researcher or graduate student, desiring to learn more about multidimensional well-being.
Jacques Silber, Emeritus Professor, Bar-Ilan University, Israel.

4) In response to the limitations of GDP as a measure of societal well-being, new indices such as the UN's Human Development Index and the OECD's Better Life Index have been developed to better capture the multidimensional nature of well-being. Chakravarty provides an accessible and lucid introduction to the theoretical literature on the multidimensional measurement of inequality, poverty, and well-being, with a particular focus on the indices that have been proposed and their axiomatic foundations. This volume is recommended to both academics and practitioners who want a state-of-the art survey of these measurement issues.
John A. Weymark, Gertrude Conaway Vanderbilt Professor of Economics and Professor of Philosophy, Vanderbilt University.

1

Well-Being as a Multidimensional Phenomenon

1.1 Introduction

The choice of income as the only attribute or dimension of well-being of a population is inappropriate since it ignores heterogeneity across individuals in many other dimensions of living conditions. Each dimension represents a particular aspect of life about which people care. Examples of such dimensions include health, literacy, and housing. A person's achievement in a dimension indicates the extent of his performance in the dimension, for instance, how healthy he is, how friendly he is, how much is his monthly income, and so on.

Only income-dependent well-being quantifiers assume that individuals with the same level of income are regarded as equally well-off irrespective of their positions in such nonincome dimensions. In their report, prepared for the Commission on the Measurement of Economic Performance and Social Progress, constituted under a French Government initiative, Stiglitz et al. (2009, p. 14) wrote "To define what wellbeing means, a multidimensional definition has to be used. Based on academic research and a number of concrete initiatives developed around the world, the Commission has identified the following key dimensions that should be taken into account. At least in principle, these dimensions should be considered simultaneously: (i) Material living standards (income, consumption and wealth); (ii) Health; (iii) Education; (iv) Personal activities including work; (v) Political voice and governance; (vi) Social connections and relationships; (vii) Environment (present and future conditions); (viii) Insecurity, of an economic as well as a physical nature. All these dimensions shape people's wellbeing, and yet many of them are missed by conventional income measures."

The need for analysis of well-being from multidimensional perspectives has also been argued in many contributions to the literature, including those of Rawls (1971); Kolm (1977); Townsend (1979); Streeten (1981); Atkinson and Bourguignon (1982); Sen (1985); Stewart (1985); Doyal and Gough (1991); Ramsay (1992); Tsui (1995); Cummins (1996); Ravallion (1996); Brandolini and D'Alessio (1998); Narayan (2000); Nussbaum (2000); Osberg and Sharpe

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(2002); Atkinson (2003); Bourguignon and Chakravarty (2003); Savaglio (2006a,b); Weymark (2006); Thorbecke (2008), Lasso de la Vega et al. (2009), Fleurbaey and Blanchet (2013); Aaberge and Brandolini (2015), Alkire et al. (2015); Duclos and Tiberti (2016).[1]

Nonmonetary dimensions of well-being are not unambiguously perfectly correlated with income. Consider a situation where, in some municipality of a developing country, there is a suboptimal supply of a local public good, say, mosquito control program. A person with a high income may not be able to trade off his income to improve his position in this nonmarketed, nonincome dimension of well-being (see Chakravarty and Lugo, 2016 and Decancq and Schokkaert, 2016).

In the capability-functioning approach, the notion of human well-being is intrinsically multidimensional (Sen, 1985, 1992; Sen and Nussbaum, 1993; Nussbaum, 2000; Pogge, 2002; Robeyns, 2009). Following John Stuart Mill, Adam Smith, and Aristotle, in the last 30 years or so, it has been reinterpreted and popularized by Sen in a series of contributions. In this approach, the traditional notions of commodity and utility are replaced respectively with functioning and capability.

Any kind of activity done or a state acquired by a person and a characteristic related to full description of the person can be regarded as a functioning. Examples include being well nourished, being healthy, being educated, and interaction with friends. Such a list can be formally represented by a vector of functionings. Capability may be defined as a set of functioning vectors that the person could have achieved.

It is possible to make a distinction between a good and functioning on the basis of operational difference. Of two persons, each owning a bicycle, the one who is physically handicapped cannot use the bike to go to the workplace as fast as the other person can. The bicycle is a good, but possessing the skill to ride it as per convenience is a functioning. This indicates that a functioning can be enacted by a good, but they are distinct concepts. Consequently, these two persons, each owning a bicycle, are not able to attain the same functioning (see Basu and López-Calva, 2011). Since the physically handicapped person, who lacks sufficient freedom to ride the bike as per desire, has a smaller capability set than the other person.

As Sen argued in several contributions, there is a clear distinction between starvation and fasting. Two persons may be in the same nutritional state, but one person fasting on some religious ground, say, is better off than the other person who is starving because he is poor. Since the former person has the freedom not to starve, his capability set is larger than that of the poor person

1 See also Clark (2016), Decancq and Neumann (2016), and Graham (2016). A recent overview of some of the related issues is available in Decancq et al. (2015).

(see also Fleurbaey, 2006a). Consequently, capabilities become closely related to freedom, opportunity, and favorable circumstances.[2]

Once the identification step, the selection of dimensions for determining human well-being, is over, at the next stage, we face the aggregation problem. The second step involves the construction of a comprehensive measure of well-being by aggregating the dimensional attainments of all individuals in the society. One simple approach can be dimension-by-dimension evaluation, resulting in a dashboard of dimensional metrics. A dashboard is a portfolio of dimension-wise well-being indictors (see Atkinson et al., 2002).[3] A dashboard can be employed to monitor each dimension in separation. But the dashboard approach has some disadvantages as well. In the words of Stiglitz et al. (2009, p. 63), "dashboards suffer because of their heterogeneity, at least in the case of very large and eclectic ones, and most lack indications about…hierarchies among the indicators used. Furthermore, as communications instruments, one frequent criticism is that they lack what has made GDP a success: the powerful attraction of a single headline figure that allows simple comparisons of socio-economic performance over time or across countries." The problem of heterogeneity across dimensional metrics can be taken care of by aggregating the dashboard-based measures into a composite index. The main disadvantage of this aggregation criterion is that it completely ignores relationships across dimensions. An alternative way to proceed toward building an all-inclusive measure of well-being is by clustering dimensional achievements across persons in terms of a real number. (See Ravallion, 2011, 2012, for a systematic comparison.)

The objective of this chapter is to evaluate how well a society is doing with respect to achievements of all the individuals in different dimensions. This is done using a social welfare function, which informs how well the society is doing when the distributions of dimensional achievements across different persons are considered. A social welfare function is regarded as a fundamental instrument in theoretical welfare economics. It has many policy-related applications. Examples include targeted equitable redistribution of income, assessment of environmental change, evaluation of health policy, cost–benefit analysis of a desired change, optimal provision of a public good, promoting goodness for future generations, assessment of legal affairs, and targeted poverty evaluation (see, among others, Balckorby et al., 2005; Adler, 2012, 2016; Boadway, 2016; Broome, 2016, and Weymark, 2016).

In order to make the chapter self-contained, in the next section, there will be a brief survey of univariate welfare measurement. Section 1.3 addresses the measurability problem of dimensional achievements. In other words, this

2 See also Qizilbash et al. (2006), Elson et al. (2011), Alkire (2016), and Krishnakumar (2007).

3 See also Slottje et al. (1991), Hicks (1997), Easterlin (2000), Hobijn and Franses (2001), Neumayer (2003), World Bank (2006), and United Nations Development Program (2005, 2010).

section clearly investigates how achievements in different dimensions can be measured. Some basics for multivariate analysis of welfare are presented in Section 1.4. The concern of Section 1.5 is the dashboard approach to the evaluation of well-being. There will be a detailed scrutiny of alternative techniques for setting weights to individual dimensional metrics. In Section 1.6, there will be an analytical discussion on axioms for a multivariate welfare function. Each axiom is a representation of a property of a welfare measure that can be defended on its own merits. Often, axioms become helpful in narrowing down the choice of welfare measures. Section 1.7 studies welfare functions, including their information requirements, which have been proposed in the literature to assess multivariate distributions of well-being. Finally, Section 1.8 concludes the discussion.

1.2 Income as a Dimension of Well-Being and Some Related Aggregations

The measurement of multidimensional welfare originates from its univariate counterpart. In consequence, a short analytical treatment of one-dimensional welfare measurement at the outset will prepare the stage for our expositions in the following sections.

It is assumed before all else that no ambiguity arises with respect to definitions and related issues of the primary elements of the analysis. For instance, should the variable on which the analysis relies be income or expenditure? How is expenditure defined? What should be the reference period of observation of incomes/expenditure? How is the threshold income that represents a minimal standard of living determined (see Chapter 2)[4]? Generally, income data are collected at the household level. Income at the individual level can be obtained from the household income by employing an appropriate equivalence scale. (See Lewbel and Pendakur, 2008, for an excellent discussion on equivalence scale.) For simplicity of exposition, we assume that the unit of analysis is "individual." If necessary, the study can be carried out at the household level.

For a population of size n, we denote an income distribution by a vector $u = (u_1, u_2, \dots , u_n) \in \mathfrak{R}^n_{++}$, where $\mathfrak{R}^n_{++}$ is the nonnegative part $\mathfrak{R}^n_+$ of the n-dimensional Euclidean space $\mathfrak{R}^n$ with the origin deleted. More precisely, $\mathfrak{R}^n_{++} = \mathfrak{R}^n_+ / \{0.1^n\}$, where 1^n is the n-coordinated vector of 1s. Here u_i stands for the income of individual i in the population. Let D^n be the positive part of $\mathfrak{R}^n_{++}$ so that $D^n = \{u \in \mathfrak{R}^n_{++} | u_i > 0 \text{ for all } i \in \{1, 2, \dots , n\}\}$. In consequence,

4 For discussion, see, among others, Anand (1983), Deaton (1992, 1997), Ravallion (1994, 1996, 2008), Deaton and Grosh (2000), World Bank (2000), Hentschel and Lanjouw (2000), Klugman (2002), Grusky and Kanbur (2006), Jenkins and Micklewright (2007), Haughton and Khandker (2009), Banerjee and Duflo (2011), Foster et al. (2013a,b), and Alkire et al. (2015).

the sets of all possible income distributions associated with $\Re^n_{++}$ and D^n become respectively $\Re_{++} = \underset{n\in N}{\cup} \Re^n_{++}$ and $D = \underset{n\in N}{\cup} D^n$, where N is a set of positive integers.

Unless stated, it will be assumed that $\Re_{++}$ represents the set of all possible income distributions. For the purpose at hand, we need to introduce some more notation. For all $n \in N$, for all $u \in \Re^n_{++}$, $\lambda(u)$ (or, simply λ) is the mean of u, $\frac{1}{n}\sum_{i=1}^{n} u_i$. For all $n \in N$, $u \in \Re^n_{++}$, let u^0 denote the nonincreasingly ordered permutation of u, that is, $u^0_1 \geq u^0_2 \geq \cdots \geq u^0_n$. Similarly, we write $\tilde{u}$ for the nondecreasingly ordered permutation of u, that is, $\tilde{u}_1 \leq \tilde{u}_2 \leq \cdots \leq \tilde{u}_n$. For all $n \in N$, for all $u, u' \in \Re^n_{++}$, we write $u \geq u'$ to mean that $u_i \geq u'_i$ for all $i \in \{1, 2, \ldots, n\}$ and $u \neq u'$. Hence, $u \geq u'$ means that at least one income in u is greater than the corresponding income in u' and no income in u is less than that in u'. The notation $u > u'$ will be used to mean that $u_i > u'_i$ for all $i \in \{1, 2, \ldots, n\}$.

An income-distribution-based social welfare function is a summary measure of the extent of well-being enjoyed by the individuals in a society, resulting from the spread of a given size of income among the individuals of the society. We denote this function by W. Formally, $W: \Re_{++} \rightarrow \Re^1_{+}$. For any $n \in N$, $u \in \Re^n_{++}$, $W(u)$ signifies the extent of welfare manifested by u. It is assumed beforehand that W is continuous so that small changes in incomes will change welfare only marginally. Since it determines the standard of welfare, we can also refer to as a welfare standard.

Next, we state certain desirable axioms for W. The terms "axiom" and "postulate" will be used interchangeably because they are assumed without proof. Each axiom represents a particular value judgment, and it may not be verifiable by factual evidence. We will as well use the terms "property" and "principle" in place of axiom. Implicit under the choice of a welfare function W is also acceptance of the axioms that are verified by W. Rawls (1971, p. 80) refers to the choice of a form W as the index problem. Since our study of their multidimensional dittos will be extensive, here our discussion will be brief.

Symmetry: For all $n \in N$, $u \in \Re^n_{++}$, $W(u) = W(\overline{u})$, where $\overline{u}$ is any reordering of u.

According to this postulate, welfare evaluation of the society remains unaffected if any two individuals swap their positions in the distribution. Equivalently, any feature other than income has no role in welfare assessment.

Symmetry Axiom for Population: For all $n \in N$, $u \in \Re^n_{++}$, $W(u) = W(u^k)$, where $u^k \in \Re^{nk}_{++}$ is the income vector in which each u_i is repeated k times, $k \geq 2$ being any positive integer.

This property, introduced by Dalton (1920), requires W to be expressed in terms of an average of the population size so that welfare judgment remains unchanged when the same population is pooled several times. It demonstrates neutrality property of the welfare standard W with respect to population size, indicating invariance of the standard under replications of the population. Consequently, the postulate becomes useful in performing comparisons of welfare across societies and of the same society over time, where the underlying population sizes are likely to differ.[5]

Increasingness: For all $n \in N$, for all $u, u' \in \Re^n_{++}$, if $u \geq u'$, then $W(u) > W(u')$.

This property claims that if at least one person's income registers an increase, then the society moves to a better welfare position. An increasing welfare function indicates preferences for higher incomes; more income is preferred to less.

The final property we wish to introduce represents equity biasness of the welfare standard. Equity orientation in welfare evaluation can be materialized through a progressive transfer, an equitable redistribution of income. Formally, for all $n \in N/\{1\}, u, u' \in \Re^n_{++,}$ we say that u is obtained from u' by a progressive transfer if for some i, j and $c > 0$ $u_i = u'_i + c \leq u_j$, $u_j = u'_j - c$, and $u_k = u'_k$ for all $k \neq i, j$. That is, u is obtained from u' by a transfer of c units of income from a person j to a person i who has lower income than j such that the transfer does not make j poorer than i and incomes of all other persons remain unaffected. Equivalently, we say that u' is obtained from u by a regressive transfer.

Pigou–Dalton Transfer: For all $n \in N/\{1\}$, for all $u, u' \in \Re^n_{++}$, if, u is obtained from u' by a progressive transfer, then $W(u) > W(u)$.

In words, welfare should increase under a progressive transfer.[6] The Pigou–Dalton transfer principle, despite its limitations, is easy to understand and becomes equivalent to several seemingly unrelated conditions. Our multidimensional dominance properties that require welfare to rise when equitable redistributions occur bear similarities with these conditions. Consequently, a discussion on these conditions becomes justifiable.

Use of a numerical example will probably make the situation clearer. Consider the ordered income vectors $u^2 = (2, 3, 4)$ and $u^1 = (1, 3, 5)$. Of these two ordered profiles, the former is obtained from the latter by a progressive

5 The term Symmetry Axiom for Population was suggested in Dasgupta et al. (1973), where overall welfare has been defined as a total concept, and replication invariance of the average welfare, overall welfare divided by the population size, was sought. Evidently, the two formulations convey the same information.

6 A limitation of a Pigou–Dalton transfer is that its size is independent of the incomes of the two affected persons. Fleurbaey and Michel (2001) suggested a proportional transfer principle where the transfer size is proportional to the incomes of the affected persons (see also Fleurbaey, 2006a). In this "leaky-bucket" transfer, the recipient receives less than what the donor transfers. A progressive transfer also disregards incomes of the persons who are richer and poorer than the donor and the recipient, respectively. For discussions on other limitations and variants of the Pigou–Dalton transfer principle, see Châteauneuf and Moyes (2006) and Chakravarty (2009, Chapter 3).

transfer of 1 unit of income from the richest person to the poorest person. This transfer does not alter the rank orders of the individuals. That is why it is a rank-preserving progressive transfer. Equivalently, we can generate u^2 by postmultiplying u^1 by some 3×3 bistochastic matrix.[7] If we denote this bistochastic matrix by B, then

$$(2,3,4) = (1,3,5)\ B = (1,3,5)\begin{pmatrix} \frac{3}{4} & 0 & \frac{1}{4} \\ 0 & 1 & 0 \\ \frac{1}{4} & 0 & \frac{3}{4} \end{pmatrix}. \quad (1.1)$$

An alternative equivalent condition for executing the redistributive operation that takes us from u^1 to u^2 is to postmultiply the former by some $n \times n$ Pigou–Dalton matrix.[8] To see this more concretely, denote the underlying Pigou–Dalton matrix by T. Then

$$(2,3,4) = (1,3,5)\ T = (1,3,5)\left[\frac{3}{4}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \frac{1}{4}\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}\right]. \quad (1.2)$$

The particular Pigou–Dalton matrix T in (1.2) is the sum of $\frac{3}{4}$ times the 3×3 identity matrix and $\frac{1}{4}$ times a 3×3 permutation matrix obtained by swapping the first and third entries in the first and third rows, respectively, of the identity matrix.

A graphical equivalence of the aforementioned three interchangeable statements is that u^2 Lorenz dominates u^1, which means that the Lorenz curve of the former in no place lies below that of the latter and lies above in some places (at least).[9] In terms of welfare ranking, this is the same as the requirement that $W(u^2) > W(u^1)$, where W is any arbitrary strictly S-concave social welfare function.[10]

7 An $n \times n$ nonnegative matrix is called a bistochastic matrix of order n if the entries in each of its rows and columns add up to 1. An $n \times n$ bistochastic matrix is called a permutation matrix if it has exactly one positive entry in each row and column.

8 A Pigou–Dalton matrix is known as a strict T-transformation in the literature. A strict T-transformation, a linear transformation defined by an $n \times n$ matrix T, is a weighted average of the $n \times n$ identity matrix and an $n \times n$ permutation matrix that just interchanges two coordinates, where the positive weights add up to 1. An $n \times n$ identity matrix is an $n \times n$ matrix whose diagonal entries are 1 and off-diagonal entries are 0 (see Marshall et al., 2011, p. 32).

9 The Lorenz curve of a nondecreasingly ordered income distribution is the graph of the cumulative proportion of the total income possessed by the bottom t proportion of the population, where t varies from 0 to 1 so that 0% of the population owns 0% of the total income and 100% of the population obtains the entire income. For an unordered or nonincreasingly ordered distribution, incomes have to be ordered nondecreasingly, and then the curve can be drawn. Upon multiplication by the mean income, the Lorenz curve of an income distribution becomes its generalized Lorenz curve.

10 A social welfare function $W: \mathfrak{R}_{++} \rightarrow \mathfrak{R}^1_+$ is called S-concave if for all $n \in N$, $u \in \mathfrak{R}^n_{++}$, and all $n \times n$ bistochastic matrices B, $W(uB) \geq W(u)$. W is called strictly S-concave, if the weak

We now review three well-known examples of univariate social welfare functions. Since multidimensional translations of these functions will be explored in detail in one of the following sections, this brief study becomes rewarding. The first example we wish to scrutinize is the symmetric mean of order $\theta(< 1)$, which for any $x \in D^n$ and $n \in N$ is defined as

$$W_A^\theta(u) = \begin{cases} \left(\frac{1}{n}\sum_{i=1}^{n} u_i^\theta\right)^{\frac{1}{\theta}}, & \theta < 1, \ \theta \neq 0, \\ \prod_{i=1}^{n} (u_i)^{\frac{1}{n}}, & \theta = 0. \end{cases} \tag{1.3}$$

Since W_A^θ is undefined for $\theta < 0$ if at least one income is nonpositive, D^n is chosen as its domain. The superscript θ in W_A^θ signifies sensitivity of the parameter θ to $W_{A,}^\theta$ and the subscript A is used to indicate that it corresponds to the Atkinson (1970) inequality index (see Chapter 2). For any $\theta \neq 0$, the aggregation process invoked in W_A^θ is as follows. First, all incomes are transformed by taking their θth power. The transformation, defined by $\left(\frac{1}{\theta}\right)$ th power of a positive real number, employed on the average $\left(\frac{1}{n}\sum_{i=1}^{n} u_i^\theta\right)$ gives us W_A^θ. This continuous, increasing, symmetric, and population-size-invariant welfare function demonstrates equity orientation (satisfaction of strict S-concavity) if and only if $\theta < 1$. Adler (2012) suggested the use of this welfare standard for moral assessment of decisions that have significant social implications.

For any income profile, an increase in the value of θ increases welfare. The reason behind this is that as the value of θ decreases, higher weights are assigned to lower incomes in the aggregation. Since the assignment of higher weights to lower income holds for all $\theta < 1$, a progressive income transfer will increase welfare by a larger amount, the lower the income of the recipient is. For $\theta = -1$, W_A^θ becomes the harmonic mean. It reduces to the geometric mean if $\theta = 0$. As $\theta \to -\infty$, W_A^θ approaches $\min_i\{u_i\}$, the maximin welfare function (Rawls, 1971), a welfare standard that prioritize the worst-off individual. In other words, in this case, welfare ranking is decided by the income of the worst-off individual.

The second welfare function we choose is the Donaldson and Weymark (1980) well-known S-Gini welfare function, which for any $u \in \Re_{++}^n$ and $n \in N$ is defined as

$$W_{DW}^\rho(u) = \frac{1}{n^\rho}\sum_{i=-1}^{n} [i^\rho - (i-1)^\rho]u_i^0. \tag{1.4}$$

inequality is replaced by a strict inequality whenever uB is not a reordering of u. For formal statements on equivalence between these conditions, see Dasgupta et al. (1973) and Marshall et al. (2011, p. 35). All S-concave functions are symmetric.

Given that incomes are nonincreasingly arranged, increasingness of the weight sequence $\{i^{\rho} - (i-1)^{\rho}\}$, where $\rho > 1$, ensures strict S-concavity (hence symmetry) of W_{DW}^{ρ}. This continuous, increasing, and population-size-invariant welfare function possesses a simple disaggregation property. If each income is broken down into two components, say, salary income and interest income, and the ranks of the individuals in the two distributions are the same, then overall welfare is simply the sum of welfares from two component distributions (see Weymark, 1981). A higher value of ρ makes welfare standard more sensitive to lower incomes within a distribution. When the single parameter ρ increases unboundedly, W_{DW}^{ρ} converges toward the maximin function. For $\rho = 2$, W_{DW}^{ρ} becomes the one-dimensional Gini welfare function $W_G(u) = \frac{1}{n^2} \sum_{i=-1}^{n} [i^2 - (i-1)^2]\, u_i^0$, a weighted average of rank-ordered incomes, where the weights themselves are rank-dependent. It is also popularly known as the Gini mean (Fleurbaey and Maniquet, 2011). Foster et al. (2013a,b) refer to this as the Sen mean.[11] It can alternatively be written as the expected value of the minimum of two randomly drawn incomes, where the random drawing is done with replacement. More precisely, $W_G(u) = \frac{1}{n^2} \sum_{i=1}^{n} \sum_{j=1}^{n} \min(u_i, u_j)$. From this formulation of the Gini mean, it is evident that for any unequal $u \in \mathfrak{R}_{++}^n$, it is less than the ordinary mean $\lambda(u)$.

Pollak (1971) analyzed the family of exponential additive welfare functions, of which a simple symmetric representation is $W_P^{\nu}(u) = -\sum_{i=-1}^{n} \exp(-\nu u_i)$, where $u \in \mathfrak{R}_+^n$ and $n \in N$ are arbitrary; $\nu > 0$ is a parameter; and "exp" stands for the exponential function. This sign restriction on $\nu > 0$ ensures that W_P^{ν} is increasing and strictly S-concave. It indicates sensitivity to lower incomes in the population. This welfare standard fails to satisfy a common property of W_A^{θ} and W_{DW}^{ρ}; if incomes are equal across individuals, welfare is judged by the equal income itself. However, the following function

$$W_{KP}^{\nu}(u) = -\frac{1}{\nu} \log\left(\frac{1}{n}\sum_{i=1}^{n} \exp(-\nu u_i)\right), \tag{1.5}$$

analyzed by Kolm (1976), which is related to W_P^{ν} via the continuous, increasing transformation $W_{KP}^{\nu}(u) = -\frac{1}{\nu} \log\left(-\frac{1}{n} W_P^{\nu}(u)\right)$, fulfills this criterion. Consequently, they will rank two income vectors over the same population in the same way. This transformation also makes W_{KP}^{ν} fulfill the symmetry postulate for population and preserves strict S-concavity (hence, symmetry) of W_P^{ν}.

11 Its first welfare theoretic axiomatic characterization was developed by Sen (1974). The characterization specifies a set of axioms for a social welfare function, which hold simultaneously if and only if the welfare function is the Sen mean. In other words, the axioms uniquely identify the Sen mean in a specific framework.

As v is increased limitlessly, W_{KP}^{v} becomes closer and closer to the maximin function.

We conclude this section by noting that while W_{A}^{θ} is linear homogeneous, W_{KP}^{v} is unit translatable. According to linear homogeneity, an equiproportionate variation in all incomes will change welfare by the same proportion. In contrast, unit translatability claims that an equal absolute change in all incomes will change welfare by the absolute amount itself.[12] An example of a linear homogeneous and unit translatable welfare function is W_{DW}^{ρ}.

1.3 Scales of Measurement: A Brief Exposition

Measurement scales specify the ways in which we can classify the variables. For each class of variables, some relevant operations can be executed so that the transmissions do not generate any loss of information (Stevens, 1946).

To grasp the issue in greater detail, suppose that w, a person's weight, is measured in kilograms. By multiplying w by 1000, we can alternatively express this weight as $w' = 1000w$ grams. This process of conversion of weight from kilograms to grams, by multiplying by the ratio $\frac{w'}{w}$, which does not lead to any loss of information on the person's weight, is admitted by indicators of ratio scale. Formally, an indicator l is said to measurable on ratio scale if there is perfect substitutability between its value v_l and cv_l, where $c > 0$ is a constant. For ratio-scale indicators, there is a natural "zero"; 0 weight means "no weight," whether it is expressed in kilograms or grams. A second example of a ratio-scale dimension is height.

An interval scale refers to a measurement in which the difference between two values can be meaningfully compared. To understand this, consider the vector of temperatures $t_C = (10, 20, 30, 40)$ expressed in degree centigrade. These temperatures can equivalently be specified in degree Fahrenheit as $t_F = (50, 68, 86, 104)$. The difference between the temperatures 20 and 10 degrees is the same as that between 40 and 30 in t_C. Similarly, there is a common difference between 68 and 50 and between 104 and 86 in t_F. The two common differences are different because the temperatures in Centigrade (C) and Fahrenheit scales (F) are connected by the one-to-one transformation $\frac{C}{5} = \frac{(F-32)}{9}$. But a temperature of 30 °C cannot be regarded as thrice as that of 10 °C. However, for a ratio-scale variable, this is meaningful. Further, there is no natural "zero" in interval scale. A 0 degree temperature does not indicate absence of heat, irrespective of whether it is stated in Centigrade or in Fahrenheit. More generally, an indicator l is said to be measurable on interval scale if its value v_l can be perfectly substituted by $a + bv_l$, where $b > 0$ and a

12 Formally, $W : \Re_{++} \rightarrow \Re_{+}^{1}$ is called linear homogeneous if for all $n \in N$, $u \in \Re_{++}^{n}$, $W(cu) = cW(u)$ for all scalars $c > 0$. Unit translatability of W requires that $W(u + c1^{n}) = W(u) + c$, where c is a scalar such that $(u + c1^{n}) \in \Re_{++}^{n}$.

are constants. A transformation of this type is called an affine transformation. A second example of an interval-scale indicator is intelligent quotient score. Variables measurable on ratio and interval scales exhaust the class of cardinally measurable variables.

A variable representing two or more mutually exclusive but not ranked categories is known as a categorical or a nominal variable. For example, we can identify female and male workers in an organization as type I and type II categories of workers. But we can as well label male workers as type I and female workers as type II workers. More precisely, there is well-defined division of the categories. Another example of a categorical variable can be labeling of subgroups of population formed by some socioeconomic characteristic, say, race, region, and religion. In contrast, for an ordinally significant variable, there is a well-defined ordering rule of the categories. For instance, we can classify individuals in a society with respect to their educational attainments into five categories: illiterate, having knowledge just to read and write in some language, elementary school graduate, high school graduate, and college graduate. We can assign the numbers 0, 1, 2, 3, and 4 to these levels of educational attainments to rank them in increasing order. Here the difference between 1 and 0 is not the same as that between 3 and 2. We can alternatively rank these categories using the numbers 0, 1, 4, 9, and 16. These numbers are obtained by squaring the previously assigned numbers 0, 1, 2, 3, and 4. Consequently, accreditation of numbers is arbitrary; the only restriction is that a higher number should be attributed to a higher category so that ranking remains preserved. Hence, the category "college graduate" should always be assigned a higher number compared to the category "high school graduate." More generally, a transformation of the type $v_l' = f(v_l)$, where f is increasing, will keep ordering of transformed values $v_l's$ of initial numbers v_ls of the variable l unaltered. Hence, any increasing function f can be regarded as an admissible transformation here. A second example of a variable with ordinal significance is "self-reported health condition," judged in terms of some health level categories, ranked in increasing order of better conditions. (See, for example, Allison and Foster (2004).) Such variables are also known as qualitative variables.[13]

1.4 Preliminaries for Multidimensional Welfare Analysis

Before we discuss the relevance of our presentation in the earlier section in the present context, let us introduce some preliminaries. We consider a

13 See, among others, Chakravarty and D'Ambrosio (2006), Jayraj and Subramanian (2009), Lasso de la Vega (2010), Aaberge and Peluso (2011), Chakravarty and Zoli (2012), Bossert et al. (2013), Aaberge and Brandolini (2014), and Alkire et al. (2015) for discussions on measurability of some socioeconomic variables that are relevant for our purpose.

society consisting of $n \in N$ individuals. Assume that there are d dimensions of well-being. The set of well-being dimensions $\{1, 2, \ldots, d\}$ is denoted by Q. The number of dimensions d is assumed to be exogenously given. Let $x_{ij} \geq 0$ stand for person i's achievement in dimension j. It is assumed at the beginning that we have complete information on these primary elements of analysis. (For social evaluations based on individuals' consumption patterns, see Jorgenson and Slesnick, 1984.)

Since $i \in \{1, 2, \ldots, n\}$ and $j \in Q$ are arbitrary, distribution of dimensional achievements in the population is represented by an $n \times d$ achievement matrix X whose (i, j)th entry is x_{ij}. The jth column of X, denoted by $x_{.j}$, shows the distribution of the total achievement $\sum_{i=1}^{n} x_{ij}$ in dimension j across n individuals. For any $j \in Q$, $\lambda(x_{.j})$ stands for the mean of the distribution $x_{.j}$. The ith row of X, denoted by $x_{i.}$, is an array of person i's achievements in different dimensions. We say that $x_{i.}$ represents person i's achievement profile in X. We will often use the terms "social matrix," "distribution matrix," and "social distribution" for an achievement matrix.

The matrix X is an arbitrary element of the set M_1^n, the set of all $n \times d$ achievement matrices with nonnegative achievements in each dimension. Let $M_2^n = \{X \in M_1^n | \lambda(x_{.j}) > 0 \quad \text{for all } j \in Q\}$. In words, M_2^n is a set of achievement matrices over the population consisting of n individuals, and the mean of achievements in each dimension is positive. Finally, define M_3^n as a set of achievement matrices over the population with size n such that for each individual, all dimensional achievements are positive. Formally, $M_3^n = \{X \in M_1^n | x_{ij} > 0 \text{ for all } i \in \{1, 2, \ldots, n\} \text{ and } j \in Q\}$. Evidently, M_1^n, M_2^n, and M_3^n can be regarded as multidimensional analogs of $\mathfrak{R}_+^n$, $\mathfrak{R}_{++}^n$, and D^n, respectively. Let M_1 stand for the set of all possible achievement matrices corresponding to M_1^n, that is, $M_1 = \bigcup_{n \in N} M_1^n$. The corresponding sets of all achievement matrices associated with M_2^n and M_3^n that parallel to M_1 are denoted respectively by M_2 and M_3. Barring anything specified, our presentation in the following sections will be made in terms of an arbitrary $M \in \{M_1, M_2, M_3\}$.

For illustrative purpose, let us assume that there are three dimensions of well-being, namely daily energy consumption in calories by an adult male,[14] per capita income, and life expectancy, measured respectively in dollars and years. With these three dimensions of well-being, we consider the following matrix X_1 as an example of an achievement matrix in a four-person economy:

$$X_1 = \begin{bmatrix} 2700 & 59.6 & 490 \\ 2500 & 65 & 900 \\ 1900 & 59.5 & 400 \\ 2700 & 62 & 600 \end{bmatrix}.$$

14 According to the US Government, an adult male requires 2700 calories per day (Public Health News, Medical News Today, 26 September 2014).