Springer INdAM Series 22

Pierluigi Colli
Angelo Favini
Elisabetta Rocca
Giulio Schimperna
Jürgen Sprekels *Editors*

Solvability, Regularity, and Optimal Control of Boundary Value Problems for PDEs

In Honour of Prof. Gianni Gilardi

Springer

Springer INdAM Series

Volume 22

More information about this series at http://www.springer.com/series/10283

Pierluigi Colli • Angelo Favini • Elisabetta Rocca •
Giulio Schimperna • Jürgen Sprekels
Editors

Solvability, Regularity, and Optimal Control of Boundary Value Problems for PDEs

In Honour of Prof. Gianni Gilardi

Springer

Editors
Pierluigi Colli
Dipartimento di Matematica
Università di Pavia
Pavia, Italy

Angelo Favini
Dipartimento di Matematica
Università di Bologna
Bologna, Italy

Elisabetta Rocca
Dipartimento di Matematica
Università di Pavia
Pavia, Italy

Giulio Schimperna
Dipartimento di Matematica
Università di Pavia
Pavia, Italy

Jürgen Sprekels
Weierstraß-Institut für Angewandte
Analysis und Stochastik (WIAS)
Berlin, Germany

Department of Mathematics
Humboldt-Universität zu Berlin
Berlin, Germany

ISSN 2281-518X ISSN 2281-5198 (electronic)
Springer INdAM Series
ISBN 978-3-319-64488-2 ISBN 978-3-319-64489-9 (eBook)
DOI 10.1007/978-3-319-64489-9

Library of Congress Control Number: 2017956717

Printed on acid-free paper

This Springer imprint is published by Springer Nature
The registered company is Springer International Publishing AG
The registered company address is: Gewerbestrasse 11, 6330 Cham, Switzerland

Preface

This special volume is dedicated to Gianni Gilardi on the occasion of his 70th birthday, in tribute to his important achievements in respect of several theoretical and applied problems, especially in the fields of partial differential equations, variational inequalities, optimal control, free boundary problems and phase transition models.

Gianni Gilardi was born in Milan in February 1947. He studied mathematics at the University of Pavia, where he graduated with full marks in October 1970. During that period, he was alumnus of the *Collegio Ghislieri*, a prestigious historical college in Pavia founded by Pope St. Pius V in 1567. After being a teaching assistant at the University of Pavia, Gianni became a full professor of mathematical analysis at the Polytechnic University of Milan in November 1980. He moved back to Pavia in 1985, where he has been appreciated as a teacher and university professor for more than 30 years. He has taught an impressive number of courses in the Schools of Engineering, Physics and Mathematics at both undergraduate and graduate levels, as well as for PhD students. He has been the advisor to a number of master and PhD students, including some of us editors of the present volume. He has never spared himself from helping colleagues, working for the community, or accepting academic responsibilities. Thus he did not hesitate in accepting the invitation to serve as chairman of the Department of Mathematics of the University of Pavia, a position he held for 6 years, or, more recently, as coordinator of the teaching programs in mathematics at the University of Pavia. In both these roles, he has been appreciated not only by colleagues but also by the administrative staff.

Gianni has been an associate fellow in the academy *Istituto Lombardo Accademia di Scienze e Lettere* since 2002. He is the author or coauthor of eight books and of around 100 research papers published in prestigious international journals. He has given numerous talks in Italy and abroad (Canada, Czech Republic, France, Germany, Japan, Portugal, Romania, Spain, Switzerland, USA) and contributed to the organization of a large number of conferences and courses.

Gianni's research activity has been intense and varied, being mainly devoted to the analysis of nonlinear PDEs, but with particular attention to the related applications. He has primarily been interested in the study of free boundary problems and phase transition models. In more detail, we may mention among his scientific interests:

– Well-posedness and regularity theory for second-order abstract evolution equations.
– Monotonicity, speed of propagation and regularity properties of the free boundary for the dam problem, the time-dependent dam problem in a general unbounded domain, and a regularity result for the time derivative of the solution.
– Error estimates for space-time discretizations of parabolic variational inequalities and a class of noncoercive stationary variational inequalities.
– Phase field models with memory and more general nonlinear Volterraintegrodifferential equations.
– Magnetostatic and electrostatic problems in inhomogeneous anisotropic media with irregular boundary and mixed boundary conditions (this includes the most cited paper coauthored by Gianni).
– Phase separation and phase segregation models including also mechanical effects.
– General phase field systems: Caginalp and Penrose–Fife models, evolutions based on the entropy balance, shape memory alloys, Cahn–Hilliard systems (also nonlocal), and dynamic boundary conditions.
– Diffuse interface models describing tumor growth dynamics.
– Control problems for phase field systems: distributed and boundary optimal control, sliding mode control, and feedback stabilization.

It is a great pleasure for us five editors of this volume to celebrate the 70th birthday of our friend Gianni. In addition to being a teacher to some of us, he has been a pleasant colleague who could always be approached with questions about mathematics or the proof of a technical lemma, knowing that he would be prepared to discuss and willing to solve problems. Gianni is very generous in providing help to young mathematicians and less young colleagues requiring his advice when checking whether "that solution" could be as regular as necessary.

His webpage contains a number of short notes, lecture notes of courses, and exercises, with examples and counterexamples that he has generously made available to students and colleagues. People who have had the chance to write papers with him experienced his generosity when, during discussions at the blackboard, they somehow began to see how the mathematical results were deduced, with Gianni already declaring his personal willingness to write down the paper.

The appreciation that Gianni always received within the scientific community is reflected in the enthusiasm with which many applied scientists and mathematicians agreed to contribute to this special volume dedicated to him, as announced in the beautiful Palazzone di Cortona during the INdAM conference "Optimal Control for Evolutionary PDEs and Related Topics" in June 2016. We editors of the present volume are warmly grateful to all the authors for their precious contributions,which will surely be appreciated also by Gianni.

The volume gathers original and peer-reviewed research papers in the field of partial differential equations, with special emphasis on mathematical models in phase transitions, complex fluids, and thermomechanics. In particular, the following thematic areas are developed: nonlinear dynamic and stationary equations, well-posedness of initial and boundary value problems for systems of PDEs, regularity properties for the solutions, optimal control problems and optimality conditions, and feedback stabilization and stability results. Most of the papers are presented in a self-contained manner; as a general strategy, the articles describe some new achievements and/or the state of the art in their line of research, providing interested readers with an overview of recent progress and future research items in PDEs.

In conclusion, we would like to join the large family of Gianni, including his wife Ce, his two daughters Carla and Laura and their husbands, his five wonderful grandchildren, his friends and the contributors to the present volume, in celebrating his accomplishments and expressing the wish that he may continue his research activity for many years to come. Let us conclude with a motto that Gianni will surely appreciate: *"Sapientia cum probitate morum coniuncta humanæ mentis perfectio"*.

Pavia, Italy — Pierluigi Colli
Bologna, Italy — Angelo Favini
Pavia, Italy — Elisabetta Rocca
Pavia, Italy — Giulio Schimperna
Berlin, Germany — Jürgen Sprekels
July 2017

Contents

About the Editors

Prof. Pierluigi Colli graduated in Mathematics at the University of Pavia in 1981, before becoming a researcher and associate professor at the same university. He became a professor of mathematical analysis at the University of Torino in 1994, and then he moved back to Pavia in 1998. He is author or coauthor of more than 150 papers, and coeditor of a number of special volumes. His main research area is the mathematical analysis of nonlinear evolution problems, in particular parabolic systems of partial differential equations arising from differential models in physics, thermodynamics, mechanics, and physiology.

Prof. Angelo Favini was an assistant professor from 1971 to 1976, and has been a professor at the University of Bologna since then. He is the author of 230 publications in international journals, mainly devoted to interpolation, differential equations in Banach spaces, partial differential equations, control theory, and ill-posed problems. His main research focus is on degenerate equations, and he has written a monograph on this subject with A. Yagi (Osaka). He is also the author of a monograph on nonlinear equations with G. Marinoschi (Bucharest), Springer-Verlag, 2012.

Prof. Elisabetta Rocca graduated in Mathematics in 1999 at the University of Pavia, where she also obtained her PhD in 2004. She was a researcher at the University of Milan till 2011 when she became associate professor. She moved to the WIAS in Berlin in 2013 where she spent 2 years coordinating a research group with the ERC Starting Grant she received as PI in 2011. She moved to the University of Pavia in 2016 where she is associate professor. She is author of more than 80 papers in mathematical analysis and applications.

Prof. Giulio Schimperna obtained his PhD in Mathematics at Milan University in 2000. Since 2006 he has been a professor of mathematical analysis in Pavia. He has authored more than 70 papers published in international scientific journals. His scientific interests mainly focus on the analysis of nonlinear evolutionary partial differential equations, and, in particular, mathematical models for phase transitions, damaging, thermomechanics, and complex fluids.

Prof. Jürgen Sprekels graduated in 1972 in Mathematics at the University of Hamburg (Germany), where he also received his PhD in 1975. He was a professor at the universities of Augsburg and Essen, before moving to a full professorship at the Humboldt-Universität zu Berlin in 1994. From 1994 to 2015, he was also the director of the Weierstrass Institute (WIAS) in Berlin. He is the coauthor of two monographs, coeditor of several conference proceedings, and coauthor of nearly 200 research papers in various fields of applied mathematics.

Rate of Convergence for Eigenfunctions of Aharonov-Bohm Operators with a Moving Pole

Laura Abatangelo and Veronica Felli

Abstract We study the behavior of eigenfunctions for magnetic Aharonov-Bohm operators with half-integer circulation and Dirichlet boundary conditions in a planar domain. We prove a sharp estimate for the rate of convergence of eigenfunctions as the pole moves in the interior of the domain.

Keywords Aharonov-Bohm potential • Convergence of eigenfunctions • Magnetic Schrödinger operators

2010 AMS Classification 35J10, 35Q40, 35J75

1 Introduction

For every $a = (a_1, a_2) \in \mathbb{R}^2$, we consider the Aharonov-Bohm vector potential with pole a and circulation $1/2$ defined as

$$A_a(x_1, x_2) = A_0(x_1 - a_1, x_2 - a_2), \quad (x_1, x_2) \in \mathbb{R}^2 \setminus \{a\},$$

L. Abatangelo
Dipartimento di Matematica e Applicazioni, Università di Milano Bicocca, Via Cozzi 55, 20125 Milano, Italy
e-mail: laura.abatangelo@unimib.it

V. Felli (✉)
Dipartimento di Scienza dei Materiali, Università di Milano Bicocca, Via Cozzi 55, 20125 Milano, Italy
e-mail: veronica.felli@unimib.it

P. Colli et al. (eds.), *Solvability, Regularity, and Optimal Control of Boundary Value Problems for PDEs*, Springer INdAM Series 22,
DOI 10.1007/978-3-319-64489-9_1

where

$$A_0(x_1,x_2) = \frac{1}{2}\left(\frac{-x_2}{x_1^2+x_2^2}, \frac{x_1}{x_1^2+x_2^2}\right), \quad (x_1,x_2) \in \mathbb{R}^2 \setminus \{(0,0)\}.$$

The Aharonov-Bohm vector potential A_a generates a δ-type magnetic field, which is called Aharonov–Bohm field: this field is produced by an infinitely long thin solenoid intersecting perpendicularly the plane (x_1,x_2) at the point a, as the radius of the solenoid tends to zero while the flux through the solenoid section remains constantly equal to $1/2$. Neglecting the irrelevant coordinate along the solenoid axis, the problem becomes 2-dimensional.

Let $\Omega \subset \mathbb{R}^2$ be a bounded, open and simply connected domain. For every $a \in \Omega$, we consider the eigenvalue problem

$$\begin{cases} (i\nabla + A_a)^2 u = \lambda u, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases} \tag{E_a}$$

in a weak sense, where the magnetic Schrödinger operator with Aharonov-Bohm potential $(i\nabla + A_a)^2$ acts on functions $u : \mathbb{R}^2 \to \mathbb{C}$ as

$$(i\nabla + A_a)^2 u = -\Delta u + 2iA_a \cdot \nabla u + |A_a|^2 u.$$

A suitable functional setting for stating a weak formulation of (E_a) can be introduced as follows: for every $a \in \Omega$, the functional space $H^{1,a}(\Omega,\mathbb{C})$ is defined as the completion of

$$\{u \in H^1(\Omega,\mathbb{C}) \cap C^\infty(\Omega,\mathbb{C}) : u \text{ vanishes in a neighborhood of } a\}$$

with respect to the norm

$$\|u\|_{H^{1,a}(\Omega,\mathbb{C})} = \left(\|(i\nabla + A_a)u\|^2_{L^2(\Omega,\mathbb{C}^2)} + \|u\|^2_{L^2(\Omega,\mathbb{C})}\right)^{1/2}.$$

In view of the following Hardy type inequality proved in [12]

$$\int_{\mathbb{R}^2} |(i\nabla + A_a)u|^2\,dx \geq \frac{1}{4}\int_{\mathbb{R}^2} \frac{|u(x)|^2}{|x-a|^2}\,dx,$$

which holds for all $a \in \mathbb{R}^2$ and $u \in C^\infty_c(\mathbb{R}^2 \setminus \{a\},\mathbb{C})$, it is easy to verify that

$$H^{1,a}(\Omega,\mathbb{C}) = \left\{u \in H^1(\Omega,\mathbb{C}) : \tfrac{u}{|x-a|} \in L^2(\Omega,\mathbb{C})\right\}.$$

We also denote as $H^{1,a}_0(\Omega,\mathbb{C})$ the space obtained as the completion of

$$C^\infty_c(\Omega \setminus \{a\},\mathbb{C})$$

with respect to the norm $\|\cdot\|_{H^{1,a}(\Omega,\mathbb{C})}$, so that

$$H_0^{1,a}(\Omega,\mathbb{C}) = \Big\{u \in H_0^1(\Omega,\mathbb{C}) : \frac{u}{|x-a|} \in L^2(\Omega,\mathbb{C})\Big\}.$$

For every $a \in \Omega$, we say that $\lambda \in \mathbb{R}$ is an eigenvalue of problem (E_a) in a weak sense if there exists $u \in H_0^{1,a}(\Omega,\mathbb{C}) \setminus \{0\}$ (called an eigenfunction) such that

$$\int_\Omega (i\nabla u + A_a u) \cdot \overline{(i\nabla v + A_a v)}\, dx = \lambda \int_\Omega u\overline{v}\, dx \quad \text{for all } v \in H_0^{1,a}(\Omega,\mathbb{C}).$$

From classical spectral theory, the eigenvalue problem (E_a) admits a sequence of real diverging eigenvalues (repeated according to their finite multiplicity)

$$\lambda_1^a \le \lambda_2^a \le \cdots \le \lambda_j^a \le \ldots.$$

The mathematical interest in Aharonov-Bohm operators with half-integer circulation can be motivated by a strong relation between spectral minimal partitions of the Dirichlet Laplacian with points of odd multiplicity and nodal domains of eigenfunctions of these operators. Indeed, a magnetic characterization of minimal partitions was given in [10] (see also [5–7, 14]): partitions with points of odd multiplicity can be obtained as nodal domains by minimizing a certain eigenvalue of an Aharonov-Bohm Hamiltonian with respect to the number and the position of poles. From this, a natural interest in the study of the properties of the map $a \mapsto \lambda_j^a$ (associating eigenvalues of magnetic operators to the position of poles) arises. In [1, 2, 4, 8, 13, 15] the behaviour of the function $a \mapsto \lambda_j^a$ in a neighborhood of a fixed point $b \in \overline{\Omega}$ has been investigated, both in the cases $b \in \Omega$ and $b \in \partial\Omega$. In particular, the analysis carried out in [1, 2, 4, 8, 15] shows that, as the pole moves towards a fixed limit pole $b \in \overline{\Omega}$, the rate of convergence of λ_j^a to λ_j^b is related to the number of nodal lines of the limit eigenfunction meeting at b. In the present paper we aim at deepening this analysis describing also the behaviour of the corresponding eigenfunctions; in particular, we will derive a sharp estimate for the rate of convergence of eigenfunctions associated to moving poles, in terms of the number of nodal lines of the limit eigenfunction.

Without loss of generality, we can assume that

$$b = 0 \in \Omega.$$

Let us assume that there exists $n_0 \ge 1$ such that

$$\lambda_{n_0}^0 \quad \text{is simple,} \tag{1}$$

and denote $\lambda_0 = \lambda_{n_0}^0$ and, for any $a \in \Omega$, $\lambda_a = \lambda_{n_0}^a$. From [13, Theorem 1.3] it follows that the map $a \mapsto \lambda_a$ is analytic in a neighborhood of 0; in particular we have that

$$\lambda_a \to \lambda_0, \quad \text{as } a \to 0. \tag{2}$$

Let $\varphi_0 \in H^{1,0}_0(\Omega, \mathbb{C}) \setminus \{0\}$ be a $L^2(\Omega, \mathbb{C})$-normalized eigenfunction of problem (E_0) associated to the eigenvalue $\lambda_0 = \lambda^0_{n_0}$, i.e. satisfying

$$\begin{cases} (i\nabla + A_0)^2 \varphi_0 = \lambda_0 \varphi_0, & \text{in } \Omega, \\ \varphi_0 = 0, & \text{on } \partial\Omega, \\ \int_\Omega |\varphi_0(x)|^2 \, dx = 1. \end{cases} \tag{3}$$

From [9, Theorem 1.3] (see also [14, Theorem 1.5]) it is known that φ_0 has at 0 a zero of order $\frac{k}{2}$ for some odd $k \in \mathbb{N}$, i.e. there exist $k \in \mathbb{N}$ odd and $\beta_1, \beta_2 \in \mathbb{C}$ such that $(\beta_1, \beta_2) \neq (0, 0)$ and

$$r^{-k/2} \varphi_0(r(\cos t, \sin t)) \to e^{i\frac{t}{2}} \left(\beta_1 \cos\left(\frac{k}{2} t\right) + \beta_2 \sin\left(\frac{k}{2} t\right) \right) \quad \text{in } C^{1,\tau}([0, 2\pi], \mathbb{C}) \tag{4}$$

as $r \to 0^+$ for any $\tau \in (0, 1)$. The asymptotics (4) (together with the fact that the right hand side of (4) is a complex multiple of a real-valued function, see [11]) implies that φ_0 has exactly k nodal lines meeting at 0 and dividing the whole angle into k equal parts; such nodal lines are tangent to the k half-lines

$$\left\{ \left(t, \tan\left(\alpha_0 + j\frac{2\pi}{k} \right) t \right) : t > 0 \right\}, \quad j = 0, 1, \dots, k-1,$$

for some angle $\alpha_0 \in [0, \frac{2\pi}{k})$.

In [1, 2] it has been proved that, under assumption (1) and being k as in (4),

$$\frac{\lambda_0 - \lambda_a}{|a|^k} \to C_0 \cos\big(k(\alpha - \alpha_0)\big) \qquad \text{as } a \to 0 \text{ with } a = |a|(\cos\alpha, \sin\alpha), \tag{5}$$

where $C_0 > 0$ is a positive constant depending only on k, β_1, and β_2. More precisely, in [1, 2] it has been proved that

$$C_0 = -4(|\beta_1|^2 + |\beta_2|^2)\, \mathfrak{m}_k$$

where

$$\mathfrak{m}_k = \min_{u \in \mathscr{D}^{1,2}_s(\mathbb{R}^2_+)} \left[\frac{1}{2} \int_{\mathbb{R}^2_+} |\nabla u(x)|^2 \, dx - \frac{k}{2} \int_0^1 t^{\frac{k}{2}-1} u(t, 0) \, dt \right] < 0. \tag{6}$$

In (6), s denotes the half-line $s := \{(x_1, x_2) \in \mathbb{R}^2 : x_2 = 0 \text{ and } x_1 \geq 1\}$ and $\mathscr{D}^{1,2}_s(\mathbb{R}^2_+)$ is the completion of $C^\infty_c(\overline{\mathbb{R}^2_+} \setminus s)$ under the norm $(\int_{\mathbb{R}^2_+} |\nabla u|^2 \, dx)^{1/2}$.

Let us now consider a suitable family of eigenfunctions relative to the approximating eigenvalue λ_a. In order to choose eigenfunctions with a suitably normalized phase, let us introduce the following notations.

For every $\alpha \in [0, 2\pi)$ and $b = (b_1, b_2) = |b|(\cos\alpha, \sin\alpha) \in \mathbb{R}^2 \setminus \{0\}$, we define

$$\theta_b : \mathbb{R}^2 \setminus \{b\} \to [\alpha, \alpha + 2\pi) \quad \text{and} \quad \theta_0^b : \mathbb{R}^2 \setminus \{0\} \to [\alpha, \alpha + 2\pi)$$

such that

$$\theta_b(b + r(\cos t, \sin t)) = t \quad \text{and} \quad \theta_0^b(r(\cos t, \sin t)) = t, \\ \text{for all } r > 0 \text{ and } t \in [\alpha, \alpha + 2\pi).$$

We also define

$$\theta_0 : \mathbb{R}^2 \setminus \{0\} \to [0, 2\pi)$$

such that

$$\theta_0(r\cos t, r\sin t) = t \quad \text{for all } r > 0 \text{ and } t \in [0, 2\pi).$$

For all $a \in \Omega$, let $\varphi_a \in H_0^{1,a}(\Omega, \mathbb{C}) \setminus \{0\}$ be an eigenfunction of problem (E_a) associated to the eigenvalue λ_a, i.e. solving

$$\begin{cases} (i\nabla + A_a)^2 \varphi_a = \lambda_a \varphi_a, & \text{in } \Omega, \\ \varphi_a = 0, & \text{on } \partial\Omega, \end{cases} \tag{7}$$

such that its modulus and phase are normalized in such a way that

$$\int_\Omega |\varphi_a(x)|^2\, dx = 1 \quad \text{and} \quad \int_\Omega e^{\frac{i}{2}(\theta_0^a - \theta_a)(x)} \varphi_a(x) \overline{\varphi_0(x)}\, dx \text{ is a positive real number,} \tag{8}$$

where φ_0 is as in (3). From (1), (2), (3), (7), (8), and standard elliptic estimates, it follows that $\varphi_a \to \varphi_0$ in $H^1(\Omega, \mathbb{C})$ and in $C^2_{\text{loc}}(\Omega \setminus \{0\}, \mathbb{C})$ and

$$(i\nabla + A_a)\varphi_a \to (i\nabla + A_0)\varphi_0 \quad \text{in } L^2(\Omega, \mathbb{C}). \tag{9}$$

The main result of the present paper establishes the sharp rate of the convergence (9).

Theorem 1 *For $\alpha \in \mathbb{R}$, $p = (\cos\alpha, \sin\alpha)$ and $a = |a|p \in \Omega$, let $\varphi_a \in H_0^{1,a}(\Omega, \mathbb{C})$ solve Eqs. (7)–(8) and $\varphi_0 \in H_0^{1,0}(\Omega, \mathbb{C})$ be a solution to (3) satisfying (1) and (4). Then there exists $\mathfrak{L}_p > 0$ such that*

$$|a|^{-k} \left\| (i\nabla + A_a)\varphi_a - e^{\frac{i}{2}(\theta_a - \theta_0^a)} (i\nabla + A_0)\varphi_0 \right\|^2_{L^2(\Omega, \mathbb{C})} \to (|\beta_1|^2 + |\beta_2|^2)\mathfrak{L}_p \tag{10}$$

as $a = |a|p \to 0$. *Moreover the function* $\alpha \mapsto \mathfrak{L}_{(\cos\alpha,\sin\alpha)}$ *is continuous, even, and periodic with period* $\frac{2\pi}{k}$.

The constant $\mathfrak{L}_p$ in Theorem 1 can be characterized as the energy of the solution of an elliptic problem with cracks (see (22)), where jumping conditions are prescribed on the segment connecting 0 and p and on the tangent to a nodal line of φ_0, see Sect. 3.

For every $\alpha \in \mathbb{R}$, let us denote as $s_\alpha = \{t(\cos\alpha, \sin\alpha) : t \geq 0\}$ the half-line with slope α. We notice that, if $a = |a|(\cos\alpha, \sin\alpha)$, then $\nabla\big(\frac{\theta_a}{2}\big) = A_a$, $\nabla\big(\frac{\theta_0^a}{2}\big) = A_0$, and $e^{-\frac{i}{2}\theta_a}$ and $e^{-\frac{i}{2}\theta_0^a}$ are smooth in $\Omega \setminus s_\alpha$. Thus

$$i\nabla_{\Omega\setminus s_\alpha}(e^{-\frac{i}{2}\theta_a}\varphi_a) = e^{-\frac{i}{2}\theta_a}(i\nabla + A_a)\varphi_a, \quad i\nabla_{\Omega\setminus s_\alpha}(e^{-\frac{i}{2}\theta_0^a}\varphi_0) = e^{-\frac{i}{2}\theta_0^a}(i\nabla + A_0)\varphi_0,$$

where $\nabla_{\Omega\setminus s_\alpha}$ is the distributional gradient in $\Omega \setminus s_\alpha$. Hence (10) can be rewritten as

$$|a|^{-k}\left\|\nabla_{\Omega\setminus s_\alpha}(e^{-\frac{i}{2}\theta_a}\varphi_a - e^{-\frac{i}{2}\theta_0^a}\varphi_0)\right\|^2_{L^2(\Omega,\mathbb{C})} \to (|\beta_1|^2 + |\beta_2|^2)\mathfrak{L}_p$$

as $a = |a|p \to 0$; thus it can be interpreted as a sharp asymptotics of the rate of convergence of the approximating eigenfunction to the limit eigenfunction in the space $\{u \in H^1(\Omega \setminus s_\alpha) : u = 0 \text{ on } \partial\Omega\}$.

The paper is organized as follows. In Sect. 2 we fix some notation and recall some known facts. In Sect. 3 we give a variational characterization of the limit profile of scaled eigenfunctions, which is used to study the properties (positivity, evenness, periodicity) of the function $p \mapsto \mathfrak{L}_p$. Finally, in Sect. 4 we prove Theorem 1, providing estimates of the energy variation first inside disks with radius $R|a|$ and then outside such disks; this latter outer estimate is performed exploiting the invertibility of an operator associated to the limit eigenvalue problem. We mention that this strategy was first developed in [3] in the context of spectral stability for varying domains, obtained by adding thin handles to a fixed limit domain.

2 Preliminaries and Some Known Facts

Through a rotation, we can easily choose a coordinate system in such a way that one nodal line of φ_0 is tangent to the x_1-axis, i.e. $\alpha_0 = 0$. In this coordinate system, we have that, letting β_1, β_2 be as in (4),

$$\beta_1 = 0. \tag{11}$$

The asymptotics of eigenvalues established in [1, 2], as well as the estimates for eigenfunctions we are going to achieve in the present paper, are based on a blow-up analysis for scaled eigenfunctions performed in [1, 2], whose main results are briefly recalled below for the sake of completeness.

For every $p \in \mathbb{R}^2$ and $r > 0$, we denote as $D_r(p)$ the disk of center p and radius r and as $D_r = D_r(0)$ the disk of center 0 and radius r. Moreover we denote, for every $r > 0$, $D_r^+ = \{(x_1, x_2) \in D_r : x_2 > 0\}$ and $D_r^- = \{(x_1, x_2) \in D_r : x_2 < 0\}$.

First of all, we observe that (4) completely describes the behaviour of φ_0 after scaling; indeed, letting

$$W_a(x) := \frac{\varphi_0(|a|x)}{|a|^{k/2}},$$

from [9, Theorem 1.3 and Lemma 6.1] we have that, under condition (11),

$$W_a \to \beta_2 e^{\frac{i}{2}\theta_0}\psi \quad \text{as } |a| \to 0 \tag{12}$$

in $H^{1,0}(D_R, \mathbb{C})$ for every $R > 1$, where $\psi : \mathbb{R}^2 \to \mathbb{R}$ is the $\frac{k}{2}$-homogeneous function (which is harmonic on $\mathbb{R}^2 \setminus \{(r, 0) : r \geq 0\}$)

$$\psi(r\cos t, r\sin t) = r^{k/2}\sin\left(\frac{k}{2}t\right), \quad r \geq 0, \quad t \in [0, 2\pi]. \tag{13}$$

For every $p \in \mathbb{R}^2$, we denote by $\mathscr{D}_p^{1,2}(\mathbb{R}^2, \mathbb{C})$ the completion of $C_c^\infty(\mathbb{R}^N \setminus \{p\}, \mathbb{C})$ with respect to the magnetic Dirichlet norm

$$\|u\|_{\mathscr{D}_p^{1,2}(\mathbb{R}^2,\mathbb{C})} := \left(\int_{\mathbb{R}^2} \left|(i\nabla + A_p)u(x)\right|^2 dx\right)^{1/2}. \tag{14}$$

Proposition 1 ([2, Proposition 4]) *Let $\alpha \in [0, 2\pi)$ and $p = (\cos\alpha, \sin\alpha)$. There exists a unique function $\Psi_p \in H^{1,p}_{\text{loc}}(\mathbb{R}^2, \mathbb{C})$ such that*

$$(i\nabla + A_p)^2\Psi_p = 0 \quad \textit{in } \mathbb{R}^2 \textit{ in a weak } H^{1,p}\textit{-sense}, \tag{15}$$

and

$$\int_{\mathbb{R}^2\setminus D_r} \left|(i\nabla + A_p)(\Psi_p - e^{\frac{i}{2}(\theta_p - \theta_0^p)}e^{\frac{i}{2}\theta_0}\psi)\right|^2 dx < +\infty, \quad \textit{for any } r > 1, \tag{16}$$

where ψ is defined in (13). Furthermore (see [9, Theorem 1.5])

$$\Psi_p - e^{\frac{i}{2}(\theta_p - \theta_0^p)}e^{\frac{i}{2}\theta_0}\psi = O(|x|^{-1/2}), \quad \textit{as } |x| \to +\infty.$$

Theorem 2 ([2, Theorem 11 and Remark 12]) *For* $\alpha \in [0, 2\pi)$,

$$p = (\cos\alpha, \sin\alpha)$$

and $a = |a|p \in \Omega$, *let* $\varphi_a \in H^{1,a}_0(\Omega, \mathbb{C})$ *solve (7)–(8) and* $\varphi_0 \in H^{1,0}_0(\Omega, \mathbb{C})$ *be a solution to* (3) *satisfying* (1), (4), *and* (11). *Let* Ψ_p *be as in Proposition 1. Then*

$$\frac{\varphi_a(|a|x)}{|a|^{k/2}} \to \beta_2 \Psi_p \quad \text{as } a = |a|p \to 0,$$

in $H^{1,p}(D_R, \mathbb{C})$ *for every* $R > 1$ *and in* $C^2_{\text{loc}}(\mathbb{R}^2 \setminus \{p\}, \mathbb{C})$.
In the sequel, we will denote

$$\tilde{\varphi}_a(x) = \frac{\varphi_a(|a|x)}{|a|^{k/2}}.$$

Sharp estimates of the energy variation under moving of poles will be derived by approximating the eigenfunction φ_a by $H^{1,0}$-functions in the less expensive way from the energetic point of view. For every $R > 2$ and $|a|$ sufficiently small, we define these approximating functions $v_{R,a}$ as follows:

$$v_{R,a} = \begin{cases} v^{ext}_{R,a}, & \text{in } \Omega \setminus D_{R|a|}, \\ v^{int}_{R,a}, & \text{in } D_{R|a|}, \end{cases}$$

where

$$v^{ext}_{R,a} := e^{\frac{i}{2}(\theta^a_0 - \theta_a)} \varphi_a \quad \text{in } \Omega \setminus D_{R|a|}$$

solves

$$\begin{cases} (i\nabla + A_0)^2 v^{ext}_{R,a} = \lambda_a v^{ext}_{R,a}, & \text{in } \Omega \setminus D_{R|a|}, \\ v^{ext}_{R,a} = e^{\frac{i}{2}(\theta^a_0 - \theta_a)} \varphi_a & \text{on } \partial(\Omega \setminus D_{R|a|}), \end{cases}$$

whereas $v^{int}_{R,a}$ is the unique solution to the problem

$$\begin{cases} (i\nabla + A_0)^2 v^{int}_{R,a} = 0, & \text{in } D_{R|a|}, \\ v^{int}_{R,a} = e^{\frac{i}{2}(\theta^a_0 - \theta_a)} \varphi_a, & \text{on } \partial D_{R|a|}. \end{cases}$$

We notice that $v_{R,a} \in H^{1,0}_0(\Omega, \mathbb{C})$ for all $R > 2$ and a sufficiently small. For all $R > 2$ and $a = |a|p \in \Omega$ with $|a|$ small, we define

$$Z^R_a(x) := \frac{v^{int}_{R,a}(|a|x)}{|a|^{k/2}}. \tag{17}$$

For all $R > 2$ and $p = (\cos\alpha, \sin\alpha)$, we also define $z_{p,R}$ as the unique solution to

$$\begin{cases} (i\nabla + A_0)^2 z_{p,R} = 0, & \text{in } D_R, \\ z_{p,R} = e^{\frac{i}{2}(\theta_0^p - \theta_p)}\Psi_p, & \text{on } \partial D_R, \end{cases} \tag{18}$$

with Ψ_p as in Proposition 1.

Lemma 1 ([2, Remark 12]; [1, Lemma 8.3]) *For $R > 2$, $\alpha \in [0, 2\pi)$,*

$$p = (\cos\alpha, \sin\alpha)$$

and $a = |a|p \in \Omega$ small, let $\varphi_a \in H_0^{1,a}(\Omega, \mathbb{C})$ solve (7)–(8), $\varphi_0 \in H_0^{1,0}(\Omega, \mathbb{C})$ be a solution to (3) satisfying (1), (4), and (11), and Z_a^R be as in (17). Then

$$Z_a^R \to \beta_2 z_{p,R} \quad \text{as } a = |a|p \to 0 \text{ in } H^{1,0}(D_R, \mathbb{C}) \text{ for every } R > 2,$$

with $z_{p,R}$ being as in (18).

3 Variational Characterization of the Limit Profile Ψ_p

In [1], the limit profile Ψ_p was constructed by solving a minimization problem in the case $p = (1, 0)$ (i.e. for poles moving tangentially to a nodal line of the limit eigenfunction); in that case the limit profile was null on a half-line. In the spirit of [4] (where poles moving towards the boundary were considered), we extend this variational construction for poles moving along a generic direction $p = (\cos\alpha, \sin\alpha)$ and construct the limit profile by solving an elliptic crack problem prescribing the jump of the solution along the segment joining 0 and p.

Let us fix $\alpha \in \big(0, 2\pi\big)$ and $p = (\cos\alpha, \sin\alpha) \in \mathbb{S}^1$. We denote by Γ_p the segment joining 0 to p, that is to say

$$\Gamma_p = \{(r\cos\alpha, r\sin\alpha) : r \in (0, 1)\}.$$

Let $s_0 = \{(x_1, 0) : \ x_1 \geq 0\}$. We introduce the trace operators

$$\gamma^{\pm} : \bigcap_{R>0} H^1(D_R^{\pm} \setminus \Gamma_p) \longrightarrow H^{1/2}_{\text{loc}}(s_0).$$

We also define $\mathscr{H}$ as the completion of

$$\mathscr{D} = \big\{u \in H^1(\mathbb{R}^2 \setminus s_0) : \gamma^+(u) + \gamma^-(u) = 0 \text{ on } s_0 \text{ and } u = 0$$
$$\text{in neighborhoods of 0 and } \infty\big\}$$

with respect to the Dirichlet norm $\left(\int_{\mathbb{R}^2\setminus s_0}|\nabla u|^2\right)^{1/2}$. In the following lemma we prove that a Hardy-type inequality can be recovered even in dimension 2, under the jump condition $\gamma^+(u)+\gamma^-(u)=0$ forced for $\mathcal{H}$-functions.

Lemma 2 *The functions in $\mathcal{D}$ satisfy the following Hardy-type inequality:*

$$\int_{\mathbb{R}^2\setminus s_0}|\nabla\varphi(x)|^2\,dx \ge \frac{1}{4}\int_{\mathbb{R}^2}\frac{|\varphi(x)|^2}{|x|^2}\,dx \quad \textit{for all } u\in\mathcal{D}.$$

Proof This is a consequence of a suitable change of gauge combined with the Hardy-type inequality for magnetic Sobolev spaces proved in [12]. For any $\varphi\in\mathcal{D}$, the function $u:=e^{\frac{i}{2}\theta_0}\varphi\in\mathcal{D}^{1,2}_0(\mathbb{R}^2,\mathbb{C})$ according to the definition of the spaces $\mathcal{D}^{1,2}_p(\mathbb{R}^2,\mathbb{C})$ given in Sect. 2 (see (14)). From the Hardy-type inequality proved in [12], it follows that

$$\int_{\mathbb{R}^2}|(i\nabla+A_0)u(x)|^2\,dx \ge \frac{1}{4}\int_{\mathbb{R}^2}\frac{|u(x)|^2}{|x|^2}\,dx.$$

Since $\nabla\left(\frac{\theta_0}{2}\right)=A_0$ and $(i\nabla+A_0)u=ie^{\frac{i}{2}\theta_0}\nabla\varphi$ in $\mathbb{R}^2\setminus s_0$, we have that

$$\int_{\mathbb{R}^2}|(i\nabla+A_0)u(x)|^2\,dx=\int_{\mathbb{R}^2\setminus s_0}|\nabla\varphi(x)|^2\,dx \quad\text{and}\quad \int_{\mathbb{R}^2}\frac{|u(x)|^2}{|x|^2}\,dx=\int_{\mathbb{R}^2}\frac{|\varphi(x)|^2}{|x|^2}\,dx,$$

thus the proof is complete.

As a direct consequence of Lemma 2, $\mathcal{H}$ can be characterized as

$$\mathcal{H}=\left\{u\in L^1_{\rm loc}(\mathbb{R}^2):\nabla_{\mathbb{R}^2\setminus s_0}u\in L^2(\mathbb{R}^2),\ \tfrac{u}{|x|}\in L^2(\mathbb{R}^2),\ \text{and } \gamma^+(u)+\gamma^-(u)=0 \text{ on } s_0\right\},$$

where $\nabla_{\mathbb{R}^2_+\setminus s_0}u$ denotes the distributional gradient of u in $\mathbb{R}^2\setminus s_0$.

For $p\ne e$ with $e=(1,0)$, we also define the space $\mathcal{H}_p$ as the completion of

$$\mathcal{D}_p=\{u\in H^1(\mathbb{R}^2\setminus(s_0\cup\Gamma_p)):\gamma^+(u)+\gamma^-(u)=0 \text{ on } s_0 \text{ and } u=0 \\ \text{in neighborhoods of 0 and } \infty\}$$

with respect to the Dirichlet norm

$$\|u\|_{\mathcal{H}_p}:=\|\nabla u\|_{L^2(\mathbb{R}^2\setminus(s_0\cup\Gamma_p))}. \tag{19}$$

In order to prove that the space $\mathcal{H}_p$ defined above is a concrete functional space, the argument performed in Lemma 2 is no more suitable, since $\mathcal{H}_p$-functions do not satisfy a Hardy inequality in the whole $\mathbb{R}^2$. We need the following two lemmas, which establish a Hardy inequality in external domains and a Poincaré inequality in D_1 for $\mathcal{H}_p$-functions.

Lemma 3 *The functions in $\mathcal{H}_p$ satisfy the following Hardy inequality in $\mathbb{R}^2 \setminus D_1$:*

$$\|\varphi\|^2_{\mathcal{H}_p} \geq \frac{1}{4} \int_{\mathbb{R}^2 \setminus D_1} \frac{|\varphi(x)|^2}{|x|^2} \, dx, \quad \textit{for all } \varphi \in \mathcal{H}_p.$$

Proof The proof follows via a change of gauge as in the proof of Lemma 2. More precisely, we notice that, for any $\varphi \in \mathcal{D}_p$, the function u defined as $u = e^{\frac{i}{2}\theta_0}\varphi$ in $\mathbb{R}^2 \setminus D_1$ and as $u(x) = u(x/|x|^2)$ in D_1 belongs to $\mathcal{D}^{1,2}_0(\mathbb{R}^2, \mathbb{C})$. From the invariance of Dirichlet magnetic norms and Hardy norms by Kelvin transform and the Hardy-type inequality of [12], it follows that

$$\begin{aligned} \|\varphi\|^2_{\mathcal{H}_p} &\geq \int_{\mathbb{R}^2 \setminus (D_1 \cup s_0)} |\nabla \varphi(x)|^2 \, dx = \frac{1}{2} \int_{\mathbb{R}^2} |(i\nabla + A_0)u(x)|^2 \, dx \\ &\geq \frac{1}{8} \int_{\mathbb{R}^2} \frac{|u(x)|^2}{|x|^2} \, dx = \frac{1}{4} \int_{\mathbb{R}^2 \setminus D_1} \frac{|\varphi(x)|^2}{|x|^2} \, dx. \end{aligned}$$

The conclusion follows by density of $\mathcal{D}_p$ in $\mathcal{H}_p$.

Lemma 4 *The functions in $\mathcal{H}_p$ satisfy the following Poincaré inequality in D_1:*

$$\|\varphi\|^2_{\mathcal{H}_p} \geq \frac{1}{6} \int_{D_1} |\varphi(x)|^2 \, dx, \quad \textit{for all } \varphi \in \mathcal{H}_p.$$

Proof From the Divergence Theorem, the Schwarz inequality and the diamagnetic inequality, it follows that, for every $u \in H^{1,0}(D_1 \setminus \Gamma_p)$,

$$\begin{aligned} 2\int_{D_1} |u|^2 \, dx &= \int_{D_1 \setminus \Gamma_p} \Big(\mathrm{div}(|u|^2 x) - 2|u|\nabla|u| \cdot x\Big) \, dx \\ &\leq \int_{\partial D_1} |u|^2 \, ds + \int_{D_1 \setminus \Gamma_p} |u|^2 \, dx + \int_{D_1 \setminus \Gamma_p} |\nabla |u||^2 \, dx \\ &\leq \int_{\partial D_1} |u|^2 \, ds + \int_{D_1} |u|^2 \, dx + \int_{D_1 \setminus \Gamma_p} |(i\nabla + A_0)u|^2 \, dx \end{aligned}$$

where, when applying the Divergence Theorem, we have use the fact that $x \cdot \nu = 0$ on both sides of Γ_p. If $\varphi \in \mathcal{D}_p$, then $u := e^{\frac{i}{2}\theta_0}\varphi \in H^{1,0}(D_1 \setminus \Gamma_p)$ and

$$(i\nabla + A_0)u = ie^{\frac{i}{2}\theta_0}\nabla\varphi \text{ in } D_1 \setminus (s_0 \cup \Gamma_p),$$

hence the previous inequality yields

$$\int_{D_1} |\varphi|^2 \, dx \leq \int_{\partial D_1} |\varphi|^2 \, ds + \int_{D_1 \setminus (s_0 \cup \Gamma_p)} |\nabla \varphi|^2 \, dx.$$

On the other hand, via the Divergence Theorem,

$$\begin{aligned}\int_{\partial D_1} |\varphi|^2 &= \int_{\partial D_1} \varphi^2 \frac{x}{|x|^2}\cdot\nu = -\int_{\mathbb{R}^2\setminus(D_1\cup s_0)} \operatorname{div}\left(\varphi^2\frac{x}{|x|^2}\right)\\ &\quad+\int_0^{+\infty}\gamma^+(\varphi^2)\frac{(s,0)}{s^2}\cdot(0,-1)\,ds\\ &\quad+\int_0^{+\infty}\gamma^-(\varphi^2)\frac{(s,0)}{s^2}\cdot(0,1)\,ds\\ &= -\int_{\mathbb{R}^2\setminus(D_1\cup s_0)} \operatorname{div}\left(\varphi^2\frac{x}{|x|^2}\right) = -2\int_{\mathbb{R}^2\setminus(D_1\cup s_0)}\varphi\nabla\varphi\cdot\frac{x}{|x|^2}\\ &\le \int_{\mathbb{R}^2\setminus(D_1\cup s_0)}|\nabla\varphi|^2+\int_{\mathbb{R}^2\setminus D_1}\frac{|\varphi|^2}{|x|^2}\le 5\|\varphi\|^2_{\mathscr{H}_p},\end{aligned}$$

where the last inequality is obtained by Lemma 3. The proof is thus complete.

As a straightforward consequence of Lemmas 3 and 4, we can characterize the space $\mathscr{H}_p$ as

$$\Big\{u\in L^1_{\mathrm{loc}}(\mathbb{R}^2):\nabla_{\mathbb{R}^2\setminus(s_0\cup\Gamma_p)}u\in L^2(\mathbb{R}^2),\ \tfrac{u}{|x|}\in L^2(\mathbb{R}^2\setminus D_1),\ u\in L^2(D_1),\ \text{and}\\ \gamma^+(u)+\gamma^-(u)=0\text{ on }s_0\Big\}.$$

The functions in $\mathscr{H}_p$ may clearly be discontinuous on Γ_p. For this reason, we introduce two trace operators. Let us consider the sets

$$U_p^+=\{(x_1,x_2)\in\mathbb{R}^2:\cos\alpha\,x_2>\sin\alpha\,x_1\}\cap(D_1\setminus s_0)$$

and

$$U_p^-=\{(x_1,x_2)\in\mathbb{R}^2:\cos\alpha\,x_2<\sin\alpha\,x_1\}\cap(D_1\setminus s_0).$$

First, for any function u defined in a neighborhood of U_p^+, respectively U_p^-, we define the restriction

$$\mathscr{R}_p^+(u)=u|_{U_p^+},\quad\text{respectively}\quad\mathscr{R}_p^-(u)=u|_{U_p^-}.$$

We observe that, since $\mathscr{R}_p^\pm$ maps $\mathscr{H}_p$ into $H^1(U_p^\pm)$ continuously, the trace operators

$$\gamma_p^\pm:\ \mathscr{H}_p\longrightarrow H^{1/2}(\Gamma_p),\quad u\longmapsto\gamma_p^\pm(u):=\mathscr{R}_p^\pm(u)|_{\Gamma_p}$$

are well defined and continuous from $\mathscr{H}_p$ to $H^{1/2}(\Gamma_p)$. Furthermore, by Sobolev trace inequalities and the Poincaré inequality of Lemma 4, it is easy to verify that the operator norm of $\gamma_p^{\pm}$ is bounded uniformly with respect to $p \in \mathbb{S}^1$, in the sense that there exists a constant $L > 0$ independent of p such that, recalling (19),

$$\|\gamma_p^{\pm}(u)\|_{H^{1/2}(\Gamma_p)} \le L\|u\|_{\mathcal{H}_p} \quad \text{for all } u \in \mathcal{H}_p. \tag{20}$$

Clearly, for a continuous function u, $\gamma_p^+(u) = \gamma_p^-(u)$.

Furthermore, let $\nu^+ = (0, -1)$ and $\nu^- = (0, 1)$ be the normal unit vectors to s_0, whereas

$$\nu_p^+ = (\sin\alpha, -\cos\alpha) \quad \text{and} \quad \nu_p^- = -\nu_p^+$$

be the normal unit vectors to Γ_p.

For every $u \in C^1(D_1 \setminus (\Gamma_p \cup s_0))$ with

$$\mathcal{R}_p^+(u) \in C^1(\overline{U_p^+} \setminus s_0) \text{ and } \mathcal{R}_p^-(u) \in C^1(\overline{U_p^-} \setminus s_0),$$

we define the normal derivatives $\frac{\partial^{\pm} u}{\partial \nu_p^{\pm}}$ on Γ_p respectively as

$$\frac{\partial^+ u}{\partial \nu_p^+} := \nabla \mathcal{R}_p^+(u) \cdot \nu_p^+ \Big|_{\Gamma_p}, \quad \text{and} \quad \frac{\partial^- u}{\partial \nu_p^-} := \nabla \mathcal{R}_p^-(u) \cdot \nu_p^- \Big|_{\Gamma_p}.$$

Analogous definitions hold for normal derivatives on s_0 (which will be denoted just as $\frac{\partial^{\pm} u}{\partial \nu^{\pm}}$).

For $p \neq e$, where $e = (1, 0)$, we consider the minimization problem for the functional $J_p : \mathcal{H}_p \to \mathbb{R}$ defined as

$$\begin{aligned} J_p(u) &= \frac{1}{2}\int_{\mathbb{R}^2\setminus(s_0\cup\Gamma_p)} |\nabla u|^2\,dx + \int_{\Gamma_p} \frac{\partial^+\psi}{\partial \nu_p^+}\gamma_p^+(u)\,ds + \int_{\Gamma_p} \frac{\partial^-\psi}{\partial \nu_p^-}\gamma_p^-(u)\,ds \\ &= \frac{1}{2}\int_{\mathbb{R}^2\setminus(s_0\cup\Gamma_p)} |\nabla u|^2\,dx + \int_{\Gamma_p} \frac{\partial^+\psi}{\partial \nu_p^+}(\gamma_p^+(u) - \gamma_p^-(u))\,ds \end{aligned} \tag{21}$$

on the set

$$\mathcal{K}_p := \{u \in \mathcal{H}_p : \ \gamma_p^+(u+\psi) + \gamma_p^-(u+\psi) = 0\}.$$

The set $\mathcal{K}_p$ is nonempty, convex and closed, the functional J_p is coercive (see (34)), so that the problem admits a unique minimum $w_p \in \mathcal{K}_p$ which is a weak solution to the problem

$$\begin{cases} -\Delta w_p = 0, & \text{in } \mathbb{R}^2 \setminus \{s_0 \cup \Gamma_p\}, \\ \gamma^+(w_p) + \gamma^-(w_p) = 0, & \text{on } s_0, \\ \gamma_p^+(w_p + \psi) + \gamma_p^-(w_p + \psi) = 0, & \text{on } \Gamma_p, \\ \dfrac{\partial^+ w_p}{\partial \nu^+} = \dfrac{\partial^- w_p}{\partial \nu^-}, & \text{on } s_0, \\ \dfrac{\partial^+ (w_p + \psi)}{\partial \nu_p^+} = \dfrac{\partial^- (w_p + \psi)}{\partial \nu_p^-}, & \text{on } \Gamma_p. \end{cases} \tag{22}$$

Remark 1 We note that the trivial function is not a solution to the problem (22), since the two jump conditions for the solution and its normal derivative on Γ_p cannot be satisfied simultaneously by the trivial function if $p \neq e$, hence $w_p \not\equiv 0$ for all $p \neq e$.

One can easily see that the function $e^{\frac{i}{2}(\theta_p - \theta_0^p)} e^{\frac{i}{2}\theta_0}(w_p + \psi)$ satisfies (15) and (16), hence by the uniqueness stated in Proposition 1 we conclude that necessarily

$$\Psi_p = e^{\frac{i}{2}(\theta_p - \theta_0^p)} e^{\frac{i}{2}\theta_0}(w_p + \psi). \tag{23}$$

On the other hand, for $p = e$, we consider the function $w_k \in \mathcal{D}_s^{1,2}(\mathbb{R}_+^2)$ defined as the unique minimizer in (6). The function w_e defined as

$$w_e(x_1, x_2) = \begin{cases} w_k(x_1, x_2), & \text{if } x_2 \geq 0, \\ w_k(x_1, -x_2), & \text{if } x_2 \leq 0, \end{cases} \tag{24}$$

satisfies

$$w_e \in \mathcal{H}_e$$

and

$$\begin{cases} -\Delta(w_e + \psi) = 0, & \text{in } \mathbb{R}^2 \setminus s, \\ \gamma^+(w_e) + \gamma^-(w_e) = 0, & \text{on } s, \\ \dfrac{\partial^+ w_e}{\partial \nu^+} = \dfrac{\partial^- w_e}{\partial \nu^-}, & \text{on } s, \end{cases} \tag{25}$$

where $s = \{(x_1, 0) : \ x_1 \geq 1\}$ and $\mathcal{H}_e$ is defined as the completion of

$$\mathcal{D}_e = \big\{u \in H^1(\mathbb{R}^2 \setminus s) : \gamma^+(u) + \gamma^-(u) = 0 \text{ on } s \text{ and } u = 0 \\ \text{in neighborhoods of } 0 \text{ and } \infty\big\}$$

with respect to the Dirichlet norm $\|\nabla u\|_{L^2(\mathbb{R}^2\setminus s)}$. One can easily see that the function $e^{\frac{i}{2}\theta_e}(w_e + \psi)$ satisfies (15) and (16) wit $p = e$ (notice that $\theta_0^e = \theta_0$), hence by the uniqueness stated in Proposition 1 we conclude that necessarily

$$\Psi_e = e^{\frac{i}{2}\theta_e}(w_e + \psi). \tag{26}$$

In [2, Proposition 14] it was proved that

$$\lim_{a=|a|p\to 0} \frac{\lambda_0 - \lambda_a}{|a|^k} = |\beta_2|^2 k \int_0^{2\pi} w_p(\cos t, \sin t) \sin\left(\frac{k}{2}t\right) dt,$$

which, combined with (5), yields

$$-4\mathfrak{m}_k \cos(k\alpha) = k \int_0^{2\pi} w_p(\cos t, \sin t) \sin\left(\frac{k}{2}t\right) dt. \tag{27}$$

The right hand side of (27) can be related to $J_p(w_p)$ as follows.

Lemma 5 *For every $p \neq e$*

$$\int_0^{2\pi} w_p(\cos t, \sin t) \sin\left(\frac{k}{2}t\right) dt = -\frac{2}{k} J_p(w_p).$$

Proof Throughout this proof, let us denote

$$\omega_p(r) := \int_0^{2\pi} w_p(r\cos t, r\sin t) \sin\left(\frac{k}{2}t\right) dt.$$

Then we have to prove that $k\omega_p(1) = -2J_p(w_p)$. Since $-\Delta w_p = 0$ in $\mathbb{R}^2 \setminus \{s_0 \cup \Gamma_p\}$, $\gamma^+(w_p) + \gamma^-(w_p) = 0$ on s_0, and $\frac{\partial^+ w_p}{\partial\nu^+} = \frac{\partial^- w_p}{\partial\nu^-}$ on s_0, by direct calculations ω_p satisfies

$$-(r^{1+k}(r^{-k/2}\omega_p(r))')' = 0, \quad \text{in } (1, +\infty).$$

Hence there exists a constant $C \in \mathbb{R}$ such that

$$r^{-k/2}\omega_p(r) = \omega_p(1) + \frac{C}{k}\left(1 - \frac{1}{r^k}\right), \quad \text{for all } r \geq 1.$$

From (23) and Proposition 1, it follows that $\omega_p(r) = O(r^{-1/2})$ as $r \to +\infty$. Hence, letting $r \to +\infty$ in the previous relation, we find $C = -k\omega_p(1)$, so that

$$\omega_p(r) = \omega_p(1) r^{-k/2}$$

for all $r \geq 1$. By taking the derivative in this relation and in the definition of ω_p, we obtain

$$-\frac{k}{2}\omega_p(1) = \int_{\partial D_1} \frac{\partial w_p}{\partial \nu}\psi\, ds.$$

Multiplying Eq. (22) by ψ and integrating by parts over $D_1 \setminus \{s_0 \cup \Gamma_p\}$, we obtain

$$\begin{aligned}\int_{D_1\setminus\{s_0\cup\Gamma_p\}} \nabla w_p \cdot \nabla\psi\, dx &= \int_{\partial D_1} \frac{\partial w_p}{\partial \nu}\psi\, ds + \int_{\Gamma_p} \left(\frac{\partial^+ w_p}{\partial \nu_p^+} + \frac{\partial^- w_p}{\partial \nu_p^-}\right)\psi\, ds \\ &= -\frac{k}{2}\omega_p(1) + \int_{\Gamma_p} \left(\frac{\partial^+ w_p}{\partial \nu_p^+} + \frac{\partial^- w_p}{\partial \nu_p^-}\right)\psi\, ds. \qquad (28)\end{aligned}$$

Testing the equation $-\Delta\psi = 0$ by w_p and integrating by parts in $D_1 \setminus \{s_0 \cup \Gamma_p\}$, we arrive at

$$\begin{aligned}\int_{D_1\setminus\{s_0\cup\Gamma_p\}} \nabla w_p \cdot \nabla\psi\, dx &= \int_{\partial D_1} \frac{\partial \psi}{\partial \nu} w_p\, ds + \int_{\Gamma_p} \frac{\partial^+ \psi}{\partial \nu_p^+}(\gamma_p^+(w_p) - \gamma_p^-(w_p))\, ds \\ &= \frac{k}{2}\omega_p(1) + \int_{\Gamma_p} \frac{\partial^+ \psi}{\partial \nu_p^+}(\gamma_p^+(w_p) - \gamma_p^-(w_p))\, ds, \qquad (29)\end{aligned}$$

where in the last step we used the fact that $\frac{\partial\psi}{\partial\nu} = \frac{k}{2}\psi$ on ∂D_1. Combining (28) and (29), we obtain

$$k\omega_p(1) = \int_{\Gamma_p} \left(\frac{\partial^+ w_p}{\partial \nu_p^+} + \frac{\partial^- w_p}{\partial \nu_p^-}\right)\psi\, ds - \int_{\Gamma_p} \frac{\partial^+ \psi}{\partial \nu_p^+}(\gamma_p^+(w_p) - \gamma_p^-(w_p))\, ds. \qquad (30)$$

On the other hand, multiplying (22) by w_p and integrating by parts over $\mathbb{R}^2\setminus\{s_0\cup\Gamma_p\}$, we obtain

$$\int_{\mathbb{R}^2\setminus\{s_0\cup\Gamma_p\}} |\nabla w_p|^2\, dx = \int_{\Gamma_p} \frac{\partial^+ w_p}{\partial \nu_p^+}\gamma_p^+(w_p)\, ds + \int_{\Gamma_p} \frac{\partial^- w_p}{\partial \nu_p^-}\gamma_p^-(w_p)\, ds.$$

At the same time, recalling the definition of J_p (21) and taking into account the latter equation we have

$$\begin{aligned}2J_p(w_p) &= \int_{\mathbb{R}^2\setminus\{s_0\cup\Gamma_p\}} |\nabla w_p|^2\, dx + 2\int_{\Gamma_p} \frac{\partial^+ \psi}{\partial \nu_p^+}\gamma_p^+(w_p)\, ds + 2\int_{\Gamma_p} \frac{\partial^- \psi}{\partial \nu_p^-}\gamma_p^-(w_p)\, ds \\ &= \int_{\Gamma_p} \frac{\partial^+ w_p}{\partial \nu_p^+}\gamma_p^+(w_p)\, ds + \int_{\Gamma_p} \frac{\partial^- w_p}{\partial \nu_p^-}\gamma_p^-(w_p)\, ds\end{aligned}$$

$$
\begin{aligned}
&+2\int_{\Gamma_p}\frac{\partial^+\psi}{\partial\nu_p^+}\gamma_p^+(w_p)\,ds+2\int_{\Gamma_p}\frac{\partial^-\psi}{\partial\nu_p^-}\gamma_p^-(w_p)\,ds\\
&=\int_{\Gamma_p}\frac{\partial^+(w_p+\psi)}{\partial\nu_p^+}\gamma_p^+(w_p)\,ds+\int_{\Gamma_p}\frac{\partial^-(w_p+\psi)}{\partial\nu_p^-}\gamma_p^-(w_p)\,ds\\
&\quad+\int_{\Gamma_p}\frac{\partial^+\psi}{\partial\nu_p^+}\gamma_p^+(w_p)\,ds+\int_{\Gamma_p}\frac{\partial^-\psi}{\partial\nu_p^-}\gamma_p^-(w_p)\,ds\\
&=\int_{\Gamma_p}\frac{\partial^+(w_p+\psi)}{\partial\nu_p^+}\gamma_p^+(w_p+\psi)\,ds+\int_{\Gamma_p}\frac{\partial^-(w_p+\psi)}{\partial\nu_p^-}\gamma_p^-(w_p+\psi)\,ds\\
&\quad+\int_{\Gamma_p}\frac{\partial^+\psi}{\partial\nu_p^+}\gamma_p^+(w_p)\,ds+\int_{\Gamma_p}\frac{\partial^-\psi}{\partial\nu_p^-}\gamma_p^-(w_p)\,ds\\
&\quad-\int_{\Gamma_p}\frac{\partial^+(w_p+\psi)}{\partial\nu_p^+}\gamma_p^+(\psi)\,ds-\int_{\Gamma_p}\frac{\partial^-(w_p+\psi)}{\partial\nu_p^-}\gamma_p^-(\psi)\,ds
\end{aligned}
$$

from which the thesis follows by comparison with (30) recalling that in the last equivalence the first term is zero by (22) and ψ is regular on Γ_p.
From the fact that w_k attains the minimum in (6) and (24) it follows easily that

$$
\mathfrak{m}_k=\frac{1}{2}\left[\frac{1}{2}\int_{\mathbb{R}^2\setminus s_0}|\nabla w_e|^2\,dx+\int_{\Gamma_e}\frac{\partial^+\psi}{\partial\nu^+}\gamma^+(w_e)\,ds+\int_{\Gamma_e}\frac{\partial^-\psi}{\partial\nu^-}\gamma^-(w_e)\,ds\right]. \tag{31}
$$

Combining (27), Lemma 5, and (31) we conclude that, for

$$
\begin{aligned}
&\frac{1}{2}\int_{\mathbb{R}^2\setminus(s_0\cup\Gamma_p)}|\nabla w_p|^2\,dx+\int_{\Gamma_p}\frac{\partial^+\psi}{\partial\nu_p^+}\gamma_p^+(w_p)\,ds+\int_{\Gamma_p}\frac{\partial^-\psi}{\partial\nu_p^-}\gamma_p^-(w_p)\,ds\\
&=\cos(k\alpha)\left[\frac{1}{2}\int_{\mathbb{R}^2\setminus s_0}|\nabla w_e|^2\,dx+\int_{\Gamma_e}\frac{\partial^+\psi}{\partial\nu^+}\gamma^+(w_e)\,ds+\int_{\Gamma_e}\frac{\partial^-\psi}{\partial\nu^-}\gamma^-(w_e)\,ds\right].
\end{aligned} \tag{32}
$$

every $p=(\cos\alpha,\sin\alpha)\in\mathbb{S}^1\setminus\{e\}$.

Lemma 6

(i) *There exists $C>0$ (independent of $p\in\mathbb{S}^1$) such that, for all $p\in\mathbb{S}^1$,*

$$
\int_{\mathbb{R}^2\setminus\Gamma_p}\left|(i\nabla+A_p)\Psi_p-e^{\frac{i}{2}(\theta_p-\theta_0^p)}e^{\frac{i}{2}\theta_0}i\nabla\psi\right|^2dx\le C. \tag{33}
$$

(ii) *If $p_n,p\in\mathbb{S}^1$ and $p_n\to p$ in $\mathbb{S}^1$, then $\Psi_{p_n}\to\Psi_p$ weakly in $H^1(D_R,\mathbb{C})$ for every $R>1$, a.e., and in $C^{0,\alpha}_{\rm loc}(\mathbb{R}^2\setminus\{p\})$.*

Proof Let us fix $q > 2$. From the continuity of the embedding $H^{1/2}(\Gamma_p) \hookrightarrow L^q(\Gamma_p)$ and (20), we have that there exists some const > 0 independent of $p \in \mathbb{S}^1$ such that, for all $u \in \mathcal{H}_p$,

$$\left|\int_{\Gamma_p} \frac{\partial^{\pm}\psi}{\partial \nu_p^{\pm}} \gamma_p^{\pm}(u)\, ds\right| = \left|\frac{k}{2}\cos\left(\frac{k}{2}\alpha\right)\int_{\Gamma_p} |x|^{\frac{k}{2}-1}\gamma_p^{\pm}(u)\, ds\right|$$
$$\leq \frac{k}{2}\||x|^{\frac{k}{2}-1}\|_{L^{q'}(\Gamma_p)}\|\gamma_p^{\pm}(u)\|_{L^q(\Gamma_p)} \leq \text{const}\, \|\gamma_p^{\pm}(u)\|_{H^{1/2}(\Gamma_p)}$$
$$\leq \text{const}\, L\|u\|_{\mathcal{H}_p}$$

and then, from the elementary inequality $ab \leq \frac{a^2}{4\varepsilon} + \varepsilon b^2$, we deduce that, for every $\varepsilon > 0$, there exists a constant $C_\varepsilon > 0$ (depending on ε but independent of p) such that, for every $u \in \mathcal{H}_p$,

$$\left|\int_{\Gamma_p} \frac{\partial^{\pm}\psi}{\partial \nu_p^{\pm}} \gamma_p^{\pm}(u)\, ds\right| \leq \varepsilon\|u\|^2_{\mathcal{H}_p} + C_\varepsilon. \tag{34}$$

From (34) and the fact that the right hand side of (32) is bounded uniformly with respect to $p \in \mathbb{S}^1$, we deduce that for any $p = (\cos\alpha, \sin\alpha) \in \mathbb{S}^1$

$$\int_{\mathbb{R}^2\setminus(s_0\cup\Gamma_p)} |\nabla w_p|^2 \leq M \tag{35}$$

for a constant $M > 0$ independent of p. Replacing (23) ((26) for $p = e$) into (35) we obtain (33).

We have that (33) together with the Hardy-type inequality of [12] implies that $\{\Psi_p\}_{p\in\mathbb{S}^1}$ is bounded in $H^1(D_R)$ and $\{A_p\Psi_p\}_{p\in\mathbb{S}^1}$ is bounded in $L^2(D_R)$ for every $R > 1$. Hence, by a diagonal process, for every sequence $p_n \to p$ in $\mathbb{S}^1$, there exist a subsequence (still denoted as p_n) and some $\Psi \in H^1_{\rm loc}(\mathbb{R}^2)$ such that Ψ_{p_n} converges to Ψ weakly in $H^1(D_R)$ and a.e. and $A_{p_n}\Psi_{p_n}$ converges to $A_p\Psi$ weakly in $L^2(D_R)$ for every $R > 1$. In particular this implies that $\Psi \in H^{1,p}_{\rm loc}(\mathbb{R}^2, \mathbb{C})$. Passing to the limit in the equation $(i\nabla + A_{p_n})^2\Psi_{p_n} = 0$, we obtain that $(i\nabla + A_p)^2\Psi = 0$. Furthermore, by weak convergences $\nabla\Psi_{p_n} \rightharpoonup \nabla\Psi$, $A_{p_n}\Psi_{p_n} \rightharpoonup A_p\Psi$ in $L^2(D_R)$ and (33), we have that, for every $R > 1$,

$$\int_{D_R\setminus D_1} \left|(i\nabla + A_p)\Psi - e^{\frac{i}{2}(\theta_p-\theta_0^p)}e^{\frac{i}{2}\theta_0}i\nabla\psi\right|^2 dx$$
$$\leq \liminf_{n\to\infty}\int_{D_R\setminus D_1} \left|(i\nabla + A_{p_n})\Psi_{p_n} - e^{\frac{i}{2}(\theta_{p_n}-\theta_0^{p_n})}e^{\frac{i}{2}\theta_0}i\nabla\psi\right|^2 dx \leq C$$