History of Mathematics Education

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Using Design Research and History to Tackle a Fundamental Problem with School Algebra

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“Responsible people have described the main goal of the first course in algebra as an understanding of the properties of a field” (Meserve & Sobel, 1964, p. 160).

“You have a danger of people being limited throughout their lives by what math they got early on—or didn’t. There’s a lot of stuff that uses Algebra 2, and students who don’t take it may be unaware that they are limiting their options later on. On the other hand, it’s much better to have someone who genuinely understands modeling and quantitative reasoning and has a feeling for statistics than someone who took an Algebra 2 class but is totally bewildered by it.”

Mark Green, quoted in Bressoud (2016, p. 1182)

“Algebra is powerful—but it can also be frightening. It demands a shift of attention from signified to signifiers. It can then become a game in which signifiers are exchanged with other signifiers. … Algebra creates an alternative world which may be under our control, but in which some people feel that nothing is real” (Tahta, 1990, p. 58).

Sinan Kanbir • M. A. (Ken) Clements • Nerida F. Ellerton

Using Design Research and History to Tackle a Fundamental Problem with School Algebra

Improving the Quality of Algebra Education at the Middle-School Level

Foreword by A. Eamonn Kelly

Springer

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Foreword

The learning of algebra has continued to attract scientific interest for over 80 years (e.g., Layton, 1932; Rittle-Johnson, Loehr, & Durkin, 2017; Stewart, 2017; Thorndike et al., 1923). Today, students' learning of algebra has implications for government policies and standards of mathematical practice (e.g., Bartell et al., 2017).

This book considers past, present and possible future aspects of an important issue in contemporary school education—specifically, if "algebra for all" is to become something more than a slogan, then what kinds of algebra, algebra instruction, and algebra learning should feature in intended, implemented, and received school mathematics curricula (Westbury, 1980)? The first three chapters are concerned with analyses of antecedents ("How did we get to where we are now?"); the "middle" chapters take up present-day key issues ("Where are we now, and what are we doing to improve the situation?"); and the two final chapters offer reflections on issues related to future research and policy ("What should we do to improve the quality of curricula and the teaching and learning of algebra in the future?").

The authors adopted a design-research approach, which had five essential elements:

1. Identifying the main problem to be investigated, and its historical, theoretical, practical, and ethical dimensions;
2. Identifying and working with key contributors;
3. Developing a research plan involving the gathering and rigorous analyses of data;
4. Planning for, implementing, and evaluating, a sequence of events based on a "plan-act-observe-reflect" action-research program;
5. Conducting, and reporting, research in a manner consistent with the study design.

Those five elements fit well with modern principles of design research. Kelly, Lesh and Baek (2008), in their *Handbook of Design Research Methods in Education,* maintained that design research in education "is directed at developing, testing, implementing, and diffusing innovative practices to move the socially constructed forms of teaching and learning from malfunction to function, or from function to excellence" (p. 3).

The design-research investigation described here incorporated three main design elements. After the research team had identified the main problem ("Why do so many middle-school and secondary-school students fail to learn algebra well?"), they considered the theoretical, practical, and ethical dimensions of that problem. Clearly, the problem is common—yet surprisingly, there is no agreement within the international mathematics education community on how it might best be solved. The researchers decided to plan, conduct and evaluate a study which would guide and illuminate the path for further studies. The second key design element was to identify and work with the key stakeholders associated with the research program. The third design element was the co-development and implementation of a research plan which would allow for rigorous gathering and analyses of relevant data—interpretation of which might generate at least a local solution to the problem**.**

The research team created a design by which both formative and summative data in qualitative and quantitative forms would be gathered and analyzed, at various stages of the project. It was also decided that this data collection and evaluation process should incorporate a plan-act-observe-reflect action research cycle (Carr & Kemmis, 2004; Kelly & Lesh, 2008), with aspects of the plan being progressively modified in light of data gathered from pencil-and-paper tests, interviews with students, observations of workshops, and student reactions to material presented in class.

As Hjalmarson and Lesh (2008) noted, the development of the product and the development of knowledge are intertwined throughout a design-research study. In this study, student knowledge, in multiple forms, was progressively developed, and monitored. That knowledge influenced later aspects of the design process and contributed to the researchers' understanding of factors influencing the teaching and learning of school algebra.

From my perspective, this book is important for three main reasons. First, the authors address a well-recognized problem in school education; second, the study engaged key teachers, students, and school administrators so that everyone was actively involved in what they deemed to be an important piece of research with likely benefit for all concerned; and last, but not least, the research exercise demonstrated the value of a design-research approach to those from other, and equally important, research traditions. I am happy to recommend this book to anyone considering adopting a design-research orientation in education studies.

References

Bartell, T., Wager, A., Edwards, A., Battey, D., Foote, M. & Spencer, J. (2017). Toward a framework for research linking equitable teaching with the Standards for Mathematical Practice. *Journal for Research in Mathematics Education, 48*(1), 7–21. doi:10.5951/jresematheduc.48.1.0007

Carr, W., & Kemmis, S. (2004). *Education, knowledge, and action research.* London, United Kingdom: Routledge.

Hjalmarson, M., & Lesh, R. A. (2008). Engineering and design research: Intersections for education research and design. In A. E. Kelly, R. A. Lesh & J. Y. Baek (Eds.), *Handbook of design research methods in education: Innovations in science, technology, engineering and mathematics learning and teaching* (pp. 96–110). New York, NY: Routledge.

Kelly, A. E., Lesh, R. A., & Baek, J. Y. (Eds.). (2008). *Handbook of design research methods in education: Innovations in science, technology, engineering and mathematics learning and teaching*. New York, NY: Routledge.

Layton, E. T. (1932). The persistence of learning in elementary algebra. *Journal of Educational Psychology*, *23*(1), 46–55. http://dx.doi.org/10.1037/h0071929

Rittle-Johnson, B., Loehr, A.M. & Durkin, K. (2017). Promoting self-explanation to improve mathematics learning: A meta-analysis and instructional design principles. *ZDM Mathematics Education* doi:10.1007/s11858-017-0834-z.

Stewart, S. (2017). *And the rest is just algebra*. Basel, Switzerland: Springer International.

Thorndike, E. L., Cobb, M. V., Orleans, J. S., Symonds, P. M., Wald, E., & Woodyard, E. (1923). *The psychology of algebra*. Macmillan.

Westbury, I. (1980). Change and stability in the curriculum: An overview of the questions. In H. G. Steiner (Ed.), *Comparative studies of mathematics curricula: Change and stability 1960–1980* (pp. 12–36). Bielefeld, Germany: Institut für Didaktik der Mathematik-Universität Bielefeld.

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Overall Book Abstract, and Individual Abstracts for the Ten Chapters of the Book

Overall Abstract

This book is unusual in that it includes both a serious historical analysis of the international history of school algebra and a description of a design-research investigation whose aim was to improve the teaching and learning of middle-school algebra. It is argued, at the outset, that it is important that strong decisions are needed to reach a situation in which most middle- and secondary- school students will learn algebra better than they do at present.

The historical analysis identified the following six purposes for school algebra:

1. School algebra as a body of knowledge which prepares students for higher mathematical and scientific studies;
2. School algebra as the study of generalized arithmetic;
3. School algebra as a gatekeeper for entry to higher studies;
4. School algebra as an integral component of mathematical modeling of real-world contexts;
5. School algebra as a vehicle for generalizing numerical, geometrical, and other mathematical structures; and
6. School algebra as the study of variables—symbols which can represent different amounts of different quantities, and relationships between those quantities.

It is claimed that although these purposes intersect, under certain conditions, they are nevertheless all conceptually separable from each other.

One of the conclusions from the historical analyses was that for 350 years there has been a disconnect between the signifiers of school algebra (its signs, symbols, and pictures) and the intended "signifieds" (the mathematical "objects" of the intended curriculum). This led the authors to adopt some of the semiotic ideas of Charles Sanders Peirce for the theoretical framework of the main study. However, it was recognized that there is a need to assist learners to link the signifiers more readily with the intended curriculum, and that recognition resulted in a decision to adopt, as part of the study frame, the "apperception" ideas of Johann Friedrich Herbart and his followers. The Herbartians argued that students' long-term memories carry "cognitive structures"—made up of verbal knowledge, skills, images, episodes (memories of relevant events), attitudes, and also idiosyncratic internal links between these components. It is the teacher's role to engineer learning environments which will help her or his students to develop rich cognitive structures which will enhance learning.

With the idea of helping teachers to create richer algebra learning environments, the theoretical position put forward by Gina Del Campo and Ken Clements which distinguished between receptive and expressive understandings, but valued both, was incorporated into the study. Workshops aimed at helping 32 seventh-grade students to enhance their cognitive structures for algebra—with respect to the associate properties for addition and multiplication and the distributive property—and to modeling real-world problems with linear sequences, were developed, trialled, and carefully implemented as part of a successful design-research, mixed-method, investigation. Details of the investigation are given. Analyses of data suggested that the investigation was sufficiently successful that it would warrant replication.

Individual Chapter Abstracts

Chapter 1: Identifying a Problem with School Algebra

Abstract: The chapter begins by presenting data which suggest that there is a longstanding, and fundamental, problem with school algebra. The problem is that many students who try hard to understand the fundamental principles of algebra, fail to do so. But that statement raises an important question—Why do so many school students find it difficult to learn the subject well? The authors of this book set out to answer that question from three different perspectives: the first relates to the question why students are asked to learn algebra. Adopting a historical method of analysis, we identify six purposes which have been offered as reasons for why school students should study algebra. The second perspective relates to theories which might help explain why so many secondary-school students do not learn algebra well. And, the third perspective offers a set of principles which might begin to provide an answer to the fundamental problem. These principles are applied in an intervention study, with seventh-grade students, which is described later in this book.

Chapter 2: Historical Reflections on How Algebra Became a Vital Component of Middle- and Secondary-School Curricula

Abstract: The chapter begins by identifying, and placing in their historical contexts, the main issues in a longstanding debate over the purposes of school algebra. The following six purposes which have been attributed to school algebra by various writers over the past three centuries are identified, and the emphases given to the purposes at different time are discussed: (a) algebra as a body of knowledge essential to higher mathematical and scientific studies, (b) algebra as generalized arithmetic, (c) algebra as a prerequisite for entry to higher studies, (d) algebra as offering a language and set of procedures for modeling real-life problems, (e) algebra as an aid to describing structural properties in elementary mathematics, and (f) algebra as a study of variables. The question is then raised, and discussed, whether school algebra represents a unidimensional trait.

Chapter 3: Framing a Classroom Intervention Study in a Middle-School Algebra Environment

Abstract: It has become a tradition in the field of mathematics education that before a researcher outlines the research design for a study he or she should outline a theoretical framework for the investigation which is about to be conducted. Then, after research questions are stated, and the design of the study is described, the investigation takes place. The data gathering, data analyses, and interpretation are guided by the theoretical framework and conclusions are couched in terms of, and seen in the light of, the theoretical framework. There are many mathematics education researchers who regard this theory-based process as sacrosanct, as absolutely essential for high-quality research. In the first part of this chapter it is argued that the traditional "theoretical-framework" process just described is flawed, that it can result in important aspects of data being overlooked, and that it can lead to incorrect, or inappropriate, conclusions being made. It is argued that the first thing which needs to be done in a mathematics education research investigation is to identify, in clearly stated terms, the problems for which solutions are to be sought. Having done that, historical frameworks—which have only occasionally been taken seriously by mathematics education researchers—

should be provided. Then, having identified the problems and having provided a historical framework, a design-research approach ought to be adopted whereby a theory, or parts of a theory, or a combination of parts of different theories, are selected as most pertinent to the problems which are to be solved. This chapter identifies three main problems: (a) "Why do so many middle-school students experience difficulty in learning algebra?" (b) "What theoretical positions might be likely to throw light on how that problem might be best solved?" (c) "In the light of answers offered for (a) and (b), what are the specific research questions for which answers will be sought in subsequent chapters of this book?"

Chapter 4: Document Analysis: The Intended CCSSM Elementary- and Middle-School Algebra Curriculum

Abstract: Having identified the main problem ("Why do so many school students find it difficult to learn school algebra well?"), and having made decisions on historical and theoretical frameworks for the study, it was important that the intended algebra content, as defined by the common-core mathematics curriculum and by the algebra content in textbooks which had previously been used by participating students, and in the textbooks being used in the seventh grade by the students, be identified and analyzed. The ensuing document analyses, presented in this chapter, revealed that the seventh-grade students might have been expected to know the associative and distributive properties for rational numbers and, given tables of values, they might have been expected to be able to identify and summarize, mathematically, the rules for uncomplicated linear sequences.

Chapter 5: Review of Pertinent Literature

Abstract: This chapter frames the main study described in this book in terms of the theoretical positions of Charles Sanders Peirce, Johann Friedrich Herbart, and Gina Del Campo and Ken Clements. Peirce's tripartite position on semiotics (featuring signifiers, interpretants, and signifieds), Herbart's theory of apperception, and Del Campo and Clements's theory of complementary receptive and expressive modes of communication, were bundled together to form a hybrid theoretical position which gave direction to the study. The chapter closes with careful statements of six research questions which emerged not only from consideration of the various literatures, but also from a knowledge of practicalities associated with the research site, from our historical analysis of the purposes of school algebra, and from our review of the literature.

Chapter 6: Research Design and Methodology

Abstract: The main study featured a mixed-method design, with complementary quantitative and qualitative data being gathered and analyzed. Since random allocation of students to two groups occurred, it was legitimate for null and research hypotheses to be formulated for the quantitative analyses, and those hypotheses are carefully defined in this chapter. One of the important challenges was to identify the population to which inferences would be made. Details relating to the development of appropriate pencil-and-paper tests and an interview protocol are also given, as are details relating to the calculation of Cohen's *d* effect sizes.

Chapter 7: Quantitative Analyses of Data

Abstract: Quantitative data from the main study are summarized and analyzed. Both the structure and modeling workshops generated statistically significant performance gains. Thus, after students had participated in both the workshops, their performances on both parts of the *Algebra Test*—that is to say, on the questions concerning structure and on the questions concerning modeling—were much improved. Cohen's *d* effect sizes for each set of workshops (the structure workshops and the modeling workshops) were large. The chapter concludes by introducing two questions. First, although the performance gains were highly *statistically* significant, and the effect sizes large, were they *educationally* significant? And, second, "What was there about the interventions which generated such apparently impressive results?".

Chapter 8: Qualitative Analyses of Data

Abstract: Qualitative data from the main study are summarized, analyzed, and interpreted from the perspective of Herbart's theory of apperception and Del Campo and Clements's theory of receptive-expression modes of communication. For many of the students, there was evidence of "significant growth," but for some, there was "no evidence." Findings from these analyses complemented and supported findings from the quantitative analyses in Chapter 7. Qualitative analyses of pre-teaching data suggested that the students remembered very little, if anything, about structures and modeling that they had previously studied—despite the fact that common-core expectations would be that they should have a strong grasp.

Chapter 9: Answers to Research Questions, and Discussion

Abstract: Answers to the six main research questions are given, and issues arising from the answers are discussed. Both the quantitative and qualitative analyses have pointed to the success of both the structure and the modeling workshops. Initially, the seventh-grade participants had very little knowledge of the associative and distributive properties—they did not know the definitions, and could not apply the properties to numerical calculations. A similar situation was true so far as modeling was concerned—whereas, initially, some students could identify recursive rules for simple linear sequences, none could identify explicit rules. Relevant algebraic conventions and language were not known. As a result of the students' active engagement in workshops in which the students learned appropriate language and conventions, and made generalizations in terms of variables, most of the participating students—but not all of them—showed strong improvement in relation to structure and modeling. Students' knowledge of definitions and skills improved, they developed appropriate imagery, and their self-confidence when asked to answer questions relating to structure and modeling improved. The results are linked to the theories of Charles Sanders Peirce, Johann Friedrich Herbart, and Gina Del Campo and Ken Clements.

Chapter 10: Postscript: Framing Research Aimed at Improving School Algebra

Abstract: This final chapter is written as a guide to persons wishing to carry out research which aims to improve middle-school students' understanding of school algebra to the point where not only will the students be able to generalize freely, but will also be able to apply the algebra that they learn. The first point made in the chapter is that *mathematics education*

researchers need to take the history of school mathematics more seriously, because the six purposes of school algebra identified in the historical analysis presented in Chapter 2 of this book were important not only in helping the research team identify the importance of language factors in school algebra, but also in designing the study which would be carried out. The second point made was that in a design-research study the theoretical frame is likely to be *not one single* theory, but a composite theory arising from a *bundle of part-theories* that are suggested by needs revealed in the historical analysis. The third, and final point is the need for mathematics education researchers to remember that*, ultimately, the aim of school mathematics is to help students learn mathematics better,* so that the students will be competent and confident to use it whenever they might need it in the future. Research designs should be such that tight assessments can be made with respect to whether the results of the studies will help educators improve the teaching and learning of algebra in schools

Preface to the Series

From the outset it was decided that the series would comprise scholarly works on a wide variety of themes, prepared by authors from around the world. We expect that authors contributing to the series will go beyond top-down approaches to history, so that emphasis will be placed on the learning, teaching, assessment and wider cultural and societal issues associated with schools (at all levels), with adults and, more generally, with the roles of mathematics within various societies. In the past, scholarly treatises on the history of mathematics education have featured strong Eurocentric/American emphases—mainly because most researchers in the field were scholars based in European or North or South American colleges or universities. It is hoped that the books in the new series will be prepared by writers from all parts of the world.

In addition to generating texts on the history of mathematics education written by authors based in various parts of the world, an important aim of the series will be to develop and report syntheses of historical research that have already been carried out with respect to important themes in mathematics education—like, for example, "Historical Perspectives on how Language Factors Influence Mathematics Teaching and Learning," and "Important Theories Which Have Influenced the Learning and Teaching of Mathematics."

The mission for the series can be summarized as:

- To make available to scholars and interested persons throughout the world the fruits of outstanding research into the history of mathematics education;
- To provide historical syntheses of comparative research on important themes in mathematics education; and
- To establish greater interest in the history of mathematics education.

We hope that the series will provide a multi-layered canvas portraying the rich details of mathematics education from the past, while at the same time presenting historical insights which can support the future. This is a canvas which can never be complete, for today's mathematics education becomes history for tomorrow. A single snapshot of mathematics education today is, by contrast with this canvas, flat and unidimensional—a mere pixel in a detailed image. We encourage readers both to explore and to contribute to the detailed image which is beginning to take shape on the canvas for this series.

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(Series Editors)

January, 2017

Preface to the Book

Recently, David M. Bressoud, a former President of the Mathematical Association of America, wrote an important, and balanced, review of the book *The Math Myth and Other Stem Delusions,* by Andrew Hacker (see, Bressoud, 2016, Hacker, 2016). In the book, Hacker argued that too many U.S. high school students study Algebra II in which they are expected to learn operations on polynomials and understand connections between their zeros and factors, construct and compare linear, quadratic and exponential models, understand the general role of functions in modeling a relationship between two quantities, and come to see trigonometric functions as models of periodic phenomena. Bressoud pointed out that an introductory knowledge and understanding of all of those topics could be useful for most people, and is essential for those who would seek a STEM career. He acknowledged, however, that a problem might arise if all secondary students were to be expected to take Algebra II. For his part, Hacker was concerned that the content of Algebra II was not suited to the present and future needs of many secondary-school students.

We believe that the purposes of algebra in middle- and secondary-school curricula have never been subjected to careful scrutiny and that, in particular, a decent history of school algebra has been lacking. In Chapter 2, in this book, we have taken up the challenge presented by that statement and have offered the beginnings of a history, written from of a global perspective, of school algebra. Six purposes of school algebra emerge from our historical analysis. When algebra first entered secondary-school curricula, in the late 1600s, the contents were chosen by high-level mathematicians, and the students were selected because they had demonstrated that they were good at arithmetic. The fact that, initially, algebra was a subject designed for just a few élite students would have enormous ramifications for the future. Content and standards of the past began to be, and continue to be, defended by those who are worried about declining standards in the schools, and by those who are very concerned to ensure that beginning college students are well prepared for rigorous mathematics courses that they might face.

This book tells of an attempt to show how the situation might be changed. First, the main problem was identified, clearly articulated, and located within the history of school education. Then, a decision was made to design a lighthouse investigation which might suggest how ordinary middle-school students can be actively engaged in the learning of important, curriculum-relevant algebra. The investigation was in the form of a design-research study whose theoretical frame emerged from a consideration of what needed to be done to solve the main problem. That theoretical frame was pieced together from three main theories—the semiotic position of Charles Sanders Peirce, the theory of apperception by Johann Friedrich Herbart, and the receptive-expression theory of lesson design by Gina Del Campo and Ken Clements. The international and historical base of the theoretical framework can be recognized by the fact that Peirce was an American philosopher who lived more than a century ago, Herbart was a German philosopher and educator who lived two centuries ago, and Del Campo and Clements developed their theoretical position in Australia in the 1980s.

The structure of this book differs from that of many other books. In the modern age, chapters of books are sold separately (usually in electronic form) and, mindful of that fact, we have prepared each chapter assuming that it might be read as a stand-alone document. Thus, at the beginning of each chapter there is an abstract and set of key words for that chapter. Occasionally, facts and important points of view mentioned in earlier chapters are repeated in the main text of a chapter—for the benefit of those without access to other chapters. At the end of each chapter we have provided a reference list for the chapter. Before the first chapter we have reproduced the abstracts of all 10 chapters as well as an "overall" abstract for the book. And, toward the end of the book we have included a "composite reference list" which brings together all the information in the reference lists for the individual chapters.

We want to thank, sincerely, Mr. X and Mr. Y, the two teachers at School W who were our partners in the main study. They were enthusiastic and always mindful of the best interests of their students. As for the students themselves, they too were wonderful participants in the study—always giving of their very best, both in class and during the research interviews. We felt very honored when Eamonn Kelly agreed to write the foreword for this book, because it was Eamonn who stimulated our initial interest in design research. Of course, we are grateful to George Seelinger, for his unwavering support, and to other colleagues within the Department of Mathematics at Illinois State University, where we were based when we carried out the main investigation described in this book. We should also express our appreciation to the numerous persons working in archives in various parts of Australia, Brunei Darussalam, Singapore, the United States of America and the United Kingdom, who helped us in the historical component of the research reported in this book. And, of course, we are extremely grateful to Melissa James and her assistant, Sara Yanny-Tillar, and all the planning and production team at Springer, for their total cooperation with us as we prepared this book.

We would be pleased to hear from anyone interested in replicating the main study, summarized in Chapters 3 through 9 of this book.

References

Bressoud, D. M. (2016). Book review: *The Math Myth and Other STEM Delusions*. *Notices of the American Mathematical Society*, *63*(10), 1181–1183.

Hacker, D. M. (2016). *The math myth and other STEM delusions.* New York, NY: The New Press.

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Chapter 1
Identifying a Problem with School Algebra

Abstract: The chapter begins by presenting data which suggest that there is a longstanding, and fundamental, problem with school algebra. The problem is that many students who try hard to understand the fundamental principles of algebra, fail to do so. But that statement raises an important question—Why do so many school students find it difficult to learn the subject well? The authors of this book answer that question from three different perspectives: the first relates to the question why students are asked to learn algebra. Adopting a historical method of analysis, we identify six purposes which have been offered as reasons for why school students should study algebra. The second perspective relates to theories which help explain why so many secondary-school students do not learn algebra well. And, the third perspective offers a set of principles which might begin to provide an answer to the fundamental problem. These principles were applied in an intervention study with seventh-grade students which is described in this book.

Keywords: Mathematics education research, History of school algebra, Student difficulties in learning algebra, Theories in mathematics education

A Fundamental Problem with School Algebra

In the opening chapter of the *First Yearbook* of the National Council of Teachers of Mathematics (hereafter "NCTM"), David Eugene Smith (1926)—a former President of the Mathematical Association of America and a prolific author of school mathematics textbooks—claimed that algebra curricula in U.S. secondary schools had improved over the period 1900–1925 and that, in particular, a large amount of "entirely useless and uninteresting work that had cumbered up the inherited course" had been "struck out" (p. 10). Smith's optimism, however, was not shared by all of the leading educators of his time. In the same *First Yearbook,* Raleigh Schorling, then NCTM's President, pointed out, for example, that Edward Lee Thorndike, a well-known education psychologist, believed that ninth-grade algebra students "had mastery of nothing whatsoever" (quoted in Schorling, 1926, p. 65).

Eighty-five years later, in NCTM's *Seventieth Yearbook,* Jeremy Kilpatrick and Andrew Izsák (2012) began their chapter titled "A History of Algebra in the School Curriculum" with the following quotation from an unnamed editorial writer of the 1930s: "It [i.e., algebra] has caused more family rows, more tears, more heartaches, and more sleepless nights than any other school subject" (p. 3). Kilpatrick and Izsák also stated that around 1910 "algebra was fast becoming a major source of failure in school" (pp. 6–7), and that during the period from 1910 to 1950 algebra enrollments in U.S. high schools fell from 57 percent to below 25 percent. In another chapter in NCTM's *Seventieth Yearbook*, Daniel Chazan (2012) commented that just 20 years ago, "algebra was seen as abstract mathematics suitable only for students who were developmentally ready and college intending" (p. 20). He added that, in fact, as late as the 1980s some U.S. students completed high school without ever having formally studied algebra.

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He attributed that situation to two factors: first, it had never been made clear why all students might benefit from studying algebra; and second, the subject was regarded, by many, as very difficult. Many other similar comments, from a wide range of sources, could be quoted here—there can be little doubt, in fact, that the longstanding problems associated with algebra in U.S. secondary schools (Cajori, 1890) persisted throughout the twentieth century (House, 1988).

Some Performance Data, and Associated Critiques of Practices in School Algebra

The difficulties that school students experienced in learning algebra in the twentieth century were certainly not confined to North America. Around 1990, Mollie MacGregor (1991) asked 45 eleventh-grade students, in Melbourne Australia—who had all studied algebra for at least four years—to write an equation relating the number of pupils in a school and the number of teachers in the school, given the following tabular information:

Number of pupils	100	200	300	400 … 1000
Number of teachers	5	10	15	20 50

MacGregor reported that only 17 of the 45 students (38%) gave a correct answer. One might say: "Surely, it should have been obvious to students who had been studying algebra for more than four years that the number of pupils (*P,* say) was equal to 20 times the number of teachers (*T* say). Hence $P = 20T$ would be an appropriate response."

MacGregor (1991) also asked 235 ninth-grade students—each of whom had been studying algebra for at least two years in secondary schools in Melbourne—to answer a set of pencil-and-paper tasks which included the following three questions:

1. "The number *y* is eight times the number *z*." Write this information in mathematical symbols.
2. *s* and *t* are numbers, *s* is 8 more than *t*. Write an equation showing the relation between *s* and *t*.
3. The Niger River in Africa is *y* metres long. The Rhine River in Europe is *z* metres long. The Niger is three times as long as the Rhine. Write an equation which shows how *y* is related to *z*.

MacGregor reported that the percentages of correct responses were 34.5% (for Question 1), 28.1% (for Question 2) and 33.2% for Question 3. MacGregor also asked 19 eleventh-grade students to attempt Question 1, and found that only 11 of them answered it correctly. With respect to Question 1, MacGregor (1991) commented: "It was expected that all, or almost all, students would get this right. Indeed, it is hard to imagine why anyone could be wrong" (p. 50).

Pongchawee Vaiyavutjamai (2004, 2006) asked 231 ninth-grade secondary-school students in Chiang Mai, Thailand, to respond to similar questions to Questions 1, 2, and 3 (shown above)—the main difference between her data and MacGregor's was that in Thailand the questions were presented in the Thai language. Analysis revealed that the students in Thailand performed at even lower levels than had the students in Melbourne, Australia. A similar finding was reported by Lim Ting Hing in his study involving tenth-grade students in Brunei Darussalam. In Lim's (2000) study the language of testing was English—which was the language of instruction, but not the first language, for the students.

There have been many other studies in which analyses of data have revealed just how difficult many students find school algebra (see, e.g., Booth, 1984, 1988; Fujii & Stephens, 2001; Hart, 1981; Kieran, 2007; Küchemann, 1981; Sfard, 1995). Certainly, the phenomenon is not confined to a few nations. For example, data from the 2011 "Trends in Mathematics and Science Study" (TIMSS) involving students from over 40 nations suggested that less than 50% of eighth-grade students worldwide would give a correct answer to the question "If t is a number between 6 and 9, then $t + 5$ is between what two numbers?" For the same 2011 TIMSS study, only 43% of the eighth-grade sample gave a correct response to a question asking them to find the value of $100 - \frac{100}{1+t}$ when t is equal to 9 (Mullis, Martin, Foy, & Arora, 2012).

Given these performance data, it becomes important to seek answers to a fundamental question—Why do so many middle-school and secondary-school students experience difficulty learning elementary algebra? We believe that it is not acceptable to adopt a head-in-the-sand attitude by asserting that such data are unimportant because the students involved were only beginning to learn algebra. If one does not expect middle-school and lower-secondary school students to learn elementary algebra well, then why require them to study it? In case the reader thinks that we are adopting an unduly negative, or "positivist," approach, it is important to state, at the outset, that there is much agreement among mathematicians and mathematics educators across the world that many secondary-school students *are* struggling to learn algebra well (see, e.g., Cai & Knuth, 2011, Ellerton & Clements, 2011; Kiang, 2012; Kieran, 2007; Kilpatrick & Izsák, 2012; Wu, 2011). Furthermore, the phenomenon is not new (see, e.g., Sfard, 1995; Porro, 1789). Indeed, in Chapter 2 of this book we shall argue that ever since the introduction of algebra into secondary-school curricula, in the seventeenth century, students have experienced difficulties with school algebra.

Not much attention will be given, in this book, to the learning of algebra by elementary-school children, or by persons studying algebra at higher-level, post-secondary-school education institutions such as universities and community colleges. Rather, our emphasis will be on algebra in middle-school and lower-secondary-school classes. In Chapter 2 we will provide an overview of historical factors which, it has been suggested, have caused school algebra to be exceedingly difficult for many middle-school and lower-secondary-school children (aged between about 10 and 15 years). We shall argue that for over three centuries most students attending algebra classes in secondary schools struggled to understand the meanings of the main symbols and signs of school algebra, and did not develop relational understandings of the mathematics being signified by those symbols and signs. An important part of our thesis will be that parents, education administrators, politicians, mathematics teachers, mathematics educators, and mathematicians have never carefully identified the dimensions of this educational problem, and that for centuries there has been an absence of scholarly historical analyses of the history of school algebra at the secondary-school level.

Historically, the move towards including algebra in secondary school mathematics curricula relied on the advice of, or textbooks of, outstanding mathematicians such as Isaac Newton, Alexis-Claude Clairaut, and Leonhard Euler. In the seventeenth, eighteenth and nineteenth centuries, most school students who studied algebra were in schools which were selective in the sense that to gain admittance to them a student needed to show evidence that he or she could do well in mathematics. An artificially high level of expectation for school algebra was thereby created. Then, in the twentieth century when, gradually, a greater proportion of children began to proceed to secondary schools, an important question arose—which students should now be expected to study algebra, and should the subject be redefined so that the new version would fit the needs and abilities of the new generation of students?

In the twentieth and twenty-first centuries, results from international comparative performance studies (such as those conducted by the International Association for the Evaluation of Educational Achievement, and the Organisation for Economic Co-operation and Development), drew attention to the high performance in school mathematics of students in Eastern Asian nations such as Japan, Korea, Singapore, Taiwan, and Hong Kong. What has not been widely recognized is that this "superior" performance of Asian nations was, arguably, a result, at least partly, of massive participation of students in those nations in a "shadow education system" by which students received, on average, many hours of tuition each week in mathematics outside of normal school hours (Bray & Kwo, 2014). For instance, Toh (2008) reported that 97% of all school students in Singapore were participating in outside-of-school tuition classes, and that mathematics was the most common subject dealt with in those classes (see, also, Bray & Lykins, 2012).

The high performance of the East Asian nations has resulted in mathematicians and education administrators in many European and American nations or states—where the extent of shadow education tutorial classes is much less than in East Asia—believing that the teaching and learning of algebra in their schools is poor. Thus, the high standards in school algebra inherited from the past, when school algebra was reserved for an elite, are not only being retained, but are being propagated as "normal" and desirable for all.

How Well Do Middle-School Teachers Understand Algebra?

A possible reason why so many beginning algebra students find it difficult to master the subject is that many of their teachers do not have strong, relational understandings of the subject and, as a result the students are not experiencing mathematically-strong teaching of the subject (Kiang, 2012; Perel & Vairo, 1967; Wu, 2011). Several research studies have generated data which support that contention. Ellerton and Clements (2011), who investigated the algebra knowledge of 328 U.S. teacher-education students who were seeking endorsement to become specialist middle-school mathematics teachers, reported data which suggested that most of the prospective teachers had retained very little of what they had been asked to learn about algebra at school. All 328 students had passed Algebra I and Algebra II in U.S. high schools, and about one-third of them had taken calculus classes. All of them had passed an elementary general mathematics course at university level. In fact, these prospective middle-school mathematics specialists were taking their last algebra course before becoming fully qualified teachers of mathematics. Yet, as entries in Table 1.1 reveal, most of the students, did not have relational understandings of elementary algebra.

Table 1.1 summarizes Ellerton and Clements's (2011) analyses of data, generated by the 328 prospective teachers, with respect to four pairs of matching tasks. For every equation there was a "matching" inequality (e.g., $x^2 > 4$ was regarded as matching $x^2 = 9$). The tasks shown in Table 1.1 were designed for the purpose of checking whether the students—who would soon be technically qualified to teach Algebra I—had learned to think holistically about the meanings of equations and inequalities. For example, to what extent would they be able to reason that the inequality $x^2 + 2 > 0$ would be true if x were to be any real number?

Table 1.1

Percentages Correct, 328 Mathematics Teacher-Education Students on Four Equation/Inequalities Pairs (Ellerton & Clements, 2011)

Equation ("State all real numbers which would make the following true.")	Acceptable Answer	Number (& %) Correct ($n = 328$)	Matching Inequality	Acceptable Answer	Number (& %) Correct ($n = 328$)
$x^2 = 9$	3, –3	74 (23%)	$x^2 > 4$	$x > 2$ or $x < -2$	16 (5%)
$x^2 + 6 = 0$	No real solution	69 (21%)	$x^2 + 2 > 0$	All real numbers	53(16%)
$4(x + 1) = 4(x - 3)$	No real solution	173 (53%)	$9(x + 1) > 9(x - 2)$	All real numbers	77(23%)
$(x - 3)(x - 2) = 0$	2, 3	194 (59%)	$(x - 3)(x - 1) > 0$	$x < 1$ or $x > 3$	2 (1%)

Handwritten reflections (submitted for an assignment titled "Where I went wrong and why") by the 328 prospective teachers in the Ellerton and Clements (2011) study confirmed that in almost all cases they had not thought about the meanings of tasks. Thus, for example, when asked which real-number values of x would make the statement $x^2 > 4$ true, only 5% of the prospective teachers gave a correct answer (see Table 1.1). The most common answer was $x > 2$, and the second most common response was $x > \pm 2$. Although 59% of the prospective teachers gave a correct response to the quadratic equation $(x - 3)(x - 2) = 0$, half of those who were correct thought that the x in $(x - 3)$ stood for "3" and, *simultaneously*, the x in $(x - 2)$ stood for "2." In their written responses to inequality tasks, such as $(x - 3)(x - 1) > 0$, and in interviews, none of the 328 students sketched a graph.

Yet, these prospective teachers had studied school algebra during the so-called "NCTM Standards" period—when meaningful learning of mathematics was supposed to have been a matter of paramount importance. In their written reflections, students typically wrote that they had "forgotten" what they had learned in school algebra classes. Even those who had studied calculus tended to make that claim. If that was indeed the case, then one must ask—what was the point of getting them to study algebra in the first place? What is even more disquieting is that our experience has been that many experienced mathematics educators do not want to hear about such data—they tend to use pejorative language such as "positivist" to describe any interpretation which suggests that the data indicate that improvement is needed. Many do not want to be reminded of Liping Ma's (1999) comparative study of the mathematical knowledge of elementary school teachers in China and the United States of America, which revealed that U.S. elementary teachers have weaker knowledge of the structures to be associated with elementary number properties than their mathematically less-qualified counterparts in China.

The Background to How and Why this Book was Written

Most of this book is concerned with describing three research studies on the teaching and learning of algebra in five middle-school classes, conducted by a team of six researchers—three school teachers and three mathematics education researchers—during the period September 2014 through March 2016. The studies took place in two schools in a midwestern state of the United States of America. So, at the outset, it is germane to ask—why then should this book be part of a Springer series on the history of mathematics education? The first three chapters of the book, and the last chapter (Chapter 10), have been written with the express purpose of answering that question.

The investigations described in this book were originally intended to be basically a study of whether seventh- and eighth-grade students in two middle schools, and their teachers, were able to cope with the algebraic demands of sessions in which the students worked on tasks which required them to apply algebraic ideas in the context of modeling, problem-solving, and algebraic structural considerations. A pilot study (Kanbir, 2014) was aimed at developing an instrument which would be suitable for use in subsequent interventions. This aspect of the study seemed to be successful in that it generated an instrument which was regarded as valid by the teacher and the three authors, and had a Cronbach-alpha reliability of 0.84.

In a second pilot study (Kanbir, 2016), the participants were the students in three seventh-grade classes, their teacher, and the three authors of this book. The teacher participated in one-on-one professional development sessions led by the three authors, pre- and post-intervention pencil-and-paper test scores were obtained for all students, and pre- and post-intervention one-on-one interviews took place with 18 selected students—six high achievers, six middle achievers, and six low achievers. For the intervention, the teacher led students through a series of questions which were not unlike those on the written tests used in the study, and the tasks used in the one-on-one interviews. Although analyses of pre- and post-intervention test and interview data revealed that after the intervention classes the students were able to give correct answers to a higher proportion of questions than before, effect sizes for the intervention were small. Furthermore, the post-intervention interview data revealed that students still did not understand, in meaningful ways, the algebra that they had been taught.

The third study is described in Chapters 4 through 9 of this book. It, too, was an intervention study. In this third study, 32 participating seventh-grade students were engaged in small-group discussions and each contributed to group presentations to the whole class. The effect sizes were large, and comparisons of pre- and post-intervention interview data indicated that much higher levels of understanding were achieved.

Overview of this Book

This book has five sections. The first three of the sections have just one chapter each.

- The first section comprises this first chapter. It introduces readers to what we regard as the basic problem of school algebra—namely, there is a need to find out why so many beginning algebra students find algebra so difficult. It also summarizes the pilot and main studies described in this book.
- The second section comprises the second chapter. It offers a summary of the history of school algebra and, in particular, identifies six purposes which have been attributed to school algebra at various times during the past 350 years. The reason for offering such a serious overview of the history of school algebra is because we believe the fundamental difficulty cannot be properly studied unless it is placed in the context of purposes which have, in the past, been associated with the teaching and learning of algebra in middle- and lower-secondary schools.
- The third section comprises Chapter 3. It discusses a design-research approach (Kelly & Lesh, 2000; Kelly, Lesh, & Baek, 2008) which was devised and employed in an

attempt to respond realistically to the fundamental problem. The issue addressed is this: What needs to be done if there is to be a good chance of solving the problem? In answer to that question it is argued that the major problem has always been one of getting students to be fluent in their receptive and expressive understandings of the major signifiers of elementary algebraic concepts and principles. Then, a summary is provided of what the present authors believe needs to be done in order that students will become comfortable and competent in using those signifiers so that they will be able to work on algebraic tasks which are deemed to be appropriate for them at their stages of mathematical development. From this description, it should be obvious to a reader steeped in the literature that a semiotic approach to the research was adopted, the object being to help learners link commonly-used "signifiers" with desired mathematical "signifieds." Not one, but three, theoretical approaches were adopted, in order to frame the research which would be carried out.

- The fourth section comprises five chapters (Chapters 4 through 8) and describes a successful intervention study in which 32 beginning students were introduced to basic concepts involving algebraic structures and mathematical modeling.
- The fifth section comprises the final two chapters in which research questions are answered and comment is made on implications of the investigation for teaching secondary-school algebra and for further research.

An unusual feature of this book is that, despite the fact that most of it is concerned with details relating to the planning, implementing, and evaluation of a mixed-method research study, it appears as part of Springer series on the history of mathematics education. The reason behind the decision to include this book in the series was that the historical analysis provided in Chapter 2 provided an important guide when decisions were being made about the design of the main study. There is a sense in which the design-research approach to the main study suggested that not one, but three, theoretical bases for the study might be appropriate—Peirce's (1931) triadic signifier-interpretant-signified theory, Herbart's (1904) theory of apperception, and Del Campo and Clements's (1987) receptive-expression theory of classroom discourse. Parts of those three theories were combined because the historical analysis suggested that, when bundled together, the hybrid theoretical position would provide the most suitable theoretical base for what was to be studied.

There is much evidence indicating that algebra is taught and learned in different ways around the world. It is wrong to think, for example, that there is a single South-East-Asian way of teaching algebra (Leung, Park, Holton, & Clarke, 2013). Even within the same school, different algebra teachers can adopt different teaching strategies. However, no matter where young people are asked to learn algebra for the first time, they will be faced with the challenge not only of learning the chief signs and conventions of school algebra, but also of connecting those signs and conventions to properties of numbers, to graphical representations, and to posing, modeling, and solving real-world problems. Those who persist will be confronted with the concept of a variable, and be expected to acquire a language which will facilitate their attempts to generalize. In that sense, learning algebra should involve more than becoming familiar with a syntax by which letters are manipulated according to well-defined rules. This book is concerned with helping middle-school students come to grips with the "essence" of school algebra—enabling them to learn, receptively and expressively, the key "signifiers" and to connect those with mathematical objects (what one might call the "signifieds"), and then be able to apply what they have learned in a range of mathematical and real-world contexts.

It is not a matter for surprise that middle-school students find it difficult to learn the syntax and semantics to be associated with the signifiers of school algebra, for it took the mathematicians of history a long time before they arrived at modern algebraic notations for what, from a historical perspective, were conceptually difficult "signifieds." It was not until the sixteenth and seventeenth centuries that the present symbols (involving $x, y, a, b, x^2, \sqrt{x}$, etc.) began to be used—before that, rhetorical and syncopated notations were adopted, even when the object was to solve relatively simple linear or quadratic equations (Cajori, 1890, 1928; Sfard, 1995). Furthermore, the algebra to be associated with negative numbers, with "imaginary" solutions to equations such as $x^2 + 1 = 0$, as well as with the concepts of real numbers, the real-number line, the Cartesian plane, the concept of a variable, and relationships between variables, was only represented in modern notations from the seventeenth century onwards (Cajori, 1890).

Thus, modern school algebra expects young children to learn, quickly, something which took mathematicians a very long time to conceptualize and notate. Seen from that vantage point, it is not at all surprising that the signifiers of school algebra, and their associated signifieds, present major pedagogical challenges to teachers, and that young learners struggle to learn "elementary" algebra. Ironically, many modern-day mathematicians are among those who seem to think that the main content and themes of "high-school algebra" should be easily acquired by most children aged between 10 and 16 years.

References

Booth, L. R. (1984). *Algebra: Children's strategies and errors.* Windsor, Berks, UK: NFER-Nelson.

Booth, L. R. (1988). Children's difficulties in beginning algebra. In A. F. Coxford & A. P. Schulte (Eds.), *The ideas of algebra: K–12, 1988 Yearbook.* Reston, VA: National Council of Teachers of Mathematics.

Bray, M., & Kwo, O. (2014). *Regulating private tutoring for public good: Policy options for supplementary education in Asia*. Hong Kong, China: Education Research Centre (CERC), UNESCO.

Bray, M., & Lykins, C. (2012). *Shadow education: Private supplementary tutoring and its implications for policy makers in Asia*. Manila, The Philippines: Asian Development Bank.

Cai, J., & Knuth, E. (2011). A global dialogue about early algebraization from multiple perspectives. In J. Cai & E. Knuth (Eds.), *Early algebraization: A global dialogue from multiple perspectives* (pp. vii–xi). Heidelberg, Germany: Springer.

Cajori, F. (1890). *The teaching and history of mathematics in the United States* (Circular of Information No. 3, 1890). Washington, DC: Bureau of Education.

Cajori, F. (1928). *A history of mathematical notations.* La Salle, IL: The Open Court Publishing Co.

Chazan, D. (2012). The shifting landscape of school algebra in the United States. In C. E. Greenes & R. Rubinstein (Eds.), *Algebra and algebraic thinking in school mathematics* (pp. 19–33). Reston, VA: National Council of Teachers of Mathematics.

Del Campo, G., & Clements, M. A. (1987). *A manual for the professional development of teachers of beginning mathematicians*. Melbourne, Australia: Association of Independent Schools of Victoria.