Computational Music Science

Gabriel Pareyon
Silvia Pina-Romero
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The Musical-Mathematical Mind

Patterns and Transformations

Springer

Computational Music Science

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To the memory of Julián Carrillo
(1875–1965) and
Alexander Grothendieck (1928–2014)

Foreword

It is my great honour and pleasure to introduce you to this book which focuses on fundamental challenges and issues in the relatively new field of Mathematical Music Theory, in turn able to be translated into computational practice.

This book, under the title *The Musical-Mathematical Mind: Patterns and Transformations*, collects the efforts of specialists who participated in the four-day International Congress on Music and Mathematics (ICMM, which took place in Puerto Vallarta, Jalisco, Mexico, November 26–29, 2014). Its contents reflect the maturing of a variety of new conceptualisations on music and mathematics. This congress was organised by the Mexican mathematicians, musicians and musicologists Octavio A. Agustín-Aquino, Juan Sebastiàn Lach Lau, Emilio Lluis-Puebla (Congress Head), Roberto Morales-Manzanares, Pablo Padilla-Longoria, and Gabriel Pareyon (Program Chair and Main Editor).

Mexican scholars have been uniquely proactive in the propagation and support of the mathematical aspects of music in theory and practice, in creativity and epistemology. Already in 2000, the First International Seminar on Mathematical Music Theory took place in Saltillo, on the occasion of the annual congress of the Mexican Mathematical Society, and the Fourth International Seminar on Mathematical Music Theory took place in Huatulco, again in Mexico, respectively organised by Lluis-Puebla, and by Agustín-Aquino.

It is remarkable that these Mexican conferences took place in the years when the Society for Mathematics and Computation in Music (SMCM) had no conference: its conferences are biannual and have taken place in the odd years since 2007. It is also remarkable because the Mexican initiative proves that there is an increasing intensity of scholarly and artistic work centred around mathematics and music. It gives us a model of how the future of this mathemusical enterprise could look.

The program of the congress in Puerto Vallarta is not only a testimony of the high level of scientific research achieved in the early years of the 21st century, it also proposed a deep spectrum of musical, mathematical, physical, and philosophical perspectives that have emerged in this field of cultural and scientific integration since its Pythagorean origins. The big difference that we observe when comparing the state of this art to the achievements in the 20th century is the involvement of advanced techniques and concepts of modern mathematics and physics, relating for example to Grothendieck's topos theory and physical string theory. It is not astonishing that the mathematician and philosopher of modern mathematics, Fernando Zalamea, has—among other authors in this book—contributed a beautiful perspective on the philosophy that lies inside the efforts to reunite mathematics with music as approaches to a unified universal knowledge.

Minneapolis, USA
January 2016

Guerino Mazzola
(ICMM 2014, Honorary President)

Preface

Proficiency and enthusiasm are gathered in this volume, as the fruit of a long-awaited conference of international specialists who devote their lives to connect, exchange and mutually involve music with mathematics and mathematics with music. We celebrate this publication at the moment of Julián Carrillo's (1875–1965) one hundred and fortieth anniversary, to whom we also dedicated a special panel (with results to be published separate from this book) during our International Congress on Music and Mathematics (ICMM) held at Puerto Vallarta, Mexico (November, 2014).

Our conference was a unique feast of mind and feelings, sound and meaning, imagination and empiricism, as the continuation and synthesis of a long tradition. The link between music and mathematics is a notorious intersection at a common origin of human civilisation embracing aesthetics, pragmatics and abstract thought. As a matter of fact, aesthetics, pragmatics and abstraction arise as human practice deeply rooted in a primary notion of repetition, rhythm, comparison, measurement, spacialization and transformation, all of them common grounds for music and mathematics.

In every part of the world, "civilisation" is a social complexity that seems to need, from its early sources, the sprout of music and mathematics. Thus, in the context of the original civilisations of Mesoamerica, music and mathematics are also strongly associated. I should mention—at least briefly—some milestones in the long history binding music and mathematics in ancient and modern Mexico: the Olmec and the Maya peoples, so admired today for their architectural, astronomical and mathematical achievements, must also be acknowledged for creating original instruments, orchestras and choirs, as well as for developing their own graphic representation of human sounds and sounds from nature. Thereafter, among the Aztec people, the patron of poetry, symmetry, music and numbers is Xochipilli-Macuilxochitl, a name that relates the number five with the symbolisation of colour, abstraction, geometry, ratio and proportion.

Later, in the Spanish colony, Sister Juana Inés de la Cruz (1651–1695) developed her own research about the connections between harmony, numbers and geometry. Even today Sor Juana's conceptualisations are still valid for the philosophical study of music, such as the study of spirals for harmonic modelling. In 19th century

Mexico, Juan N. Adorno (1807–1880) published his treatise *Harmony of the Universe*, based on principles of physical and mathematical harmony. Later, the Porfirian thinker Juan N. Cordero (1851–1916) in his book *Examen de los acordes de transformación tonal* (*Examination of the Chords of Tonal Transformation*) proposed a principle of musical transformation based on logical axioms. A few decades after, in the 20th century, José Vasconcelos (1882–1959) claimed that "only the musical study of mathematics, and the rhythmic comprehension of numbers, could be useful as effective forms of thought and discovery of the human nature". In the same epoch, another Mexican thinker, Samuel Ramos (1882–1959) wrote that "All kinds of perturbation in the Universe are of a rhythmic nature. The fluency of changes cannot be unarticulated among them; therefore the rhythm of changes is accumulative". Quoting Sor Juana, Adorno, Cordero, Vasconcelos, and Ramos are part of what semiotician Mauricio Beuchot (1950–) —a contemporary of us—acknowledges as "the Mexican devotion of Pythagoreanism and related doctrines".

Indeed, the orientation of Mexican cultures seems to be magnetised by the intuitions of ratio, proportion, analogy, metonymy, and geometrical and algebraic transformation. We may trace this influence in the most famous composers and music theorists of modern Mexico, namely Augusto Novaro, Conlon Nancarrow, Ervin Wilson, Julio Estrada, Manuel Enríquez, Antonio Russek, Roberto Morales-Manzanares, Víctor Rasgado, and Hebert Vázquez, among others. Indeed, they influence nowadays Mexican studies on music and mathematics as a new mixed discipline. This transdisciplinarity also flourished thanks to the effort of mathematician Prof. Emilio Lluis-Puebla, who graduated an internationally active group of specialists.

As I mentioned before, our meeting also devoted a special panel to the discussion of mathematics applied to music, in honour of the great violinist, conductor, composer and maker of new musical instruments, Julián Carrillo, who through a long and very productive life achieved the invention of music that transcended the traditional Western principles of consonance and harmony, as he foresaw a "universe of endless musical scales and chords". Carrillo's project in the domain of physics and mathematics, and its musical output, is an inspiration for current discussion on these subjects, addressed from different viewpoints during our congress.

We may mention some recurring concepts and theoretical approaches that motivated us during our meeting: tessellation in topological-musical spaces, scaling and even distribution, diatonicity, algebraic transformations, networks and geometry, partitions and well-formedness theory, theories of gestures, morphisms, set theory and fuzzy logic, as well as a new discussion on elementary particles and quantum symmetry as interests of systematic musicology. Despite this variety, all our mathematical proposals fell into five general areas: I. Dynamical Systems, II. Logic, Algebra and Algorithmics, III. Gestural Theories, IV. New Methods for Music Analysis, and V. Modern Geometry and Topology. Although we followed this thematic division during our congress, this book is classified by alphabetical order of authors, for the sake of practical consultation and because most of the contributions present developments in more than one subject.

I wish to end this Preface emphasising the fact that the international President of our Congress, Prof. Guerino Mazzola, is one of the leading thinkers in the field of the Mathematical Theory of Music; and our national Head of Congress, Prof. Emilio Lluis-Puebla pioneered systematic musicology in Mexico and Latin America, organising the Seminars on Mathematical Theory of Music in previous years. We completed our group of national and international guests with the best and more original proposals received after almost two years of organisation that reached its climax during the four days of ICMM 2014. We remain grateful to all our contributors.

Guadalajara, Mexico
December 2015

Gabriel Pareyon
(ICMM 2014, Program Chair and Editor)

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This book would not have been possible without the generous support of many wonderful people and organisations. We would like to thank all authors contributing their knowledge and expertise during our International Congress on Music and Mathematics (ICMM), held in Puerto Vallarta, Mexico, November 26–29, 2014.

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Our ICMM would neither have been possible without the efforts of Dr. Yael Bitran-Goren, head of the National Centre for Music Investigation, Documentation and Information (CENIDIM–INBA, Mexico), who provided institutional support and kept faith with our project.

We also owe our gratitude to our Scientific—Organizing Committee, and to Silvia Pina-Romero, Ph.D. Math., for her generous assistance. We also thank our panel chairs' generous cooperation (R. Brotbeck, D. Clampitt, J.S. Lach-Lau, M. Montiel, S. Pina-Romero).

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Finally we want to express our most special gratitude to Clarence Barlow, D. Gareth Loy, Guerino Mazzola, and Fernando Zalamea, whose passion and wisdom connecting music and mathematics has been luminous for most of us. Our congress in Puerto Vallarta would not have been possible without your inspiration.

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Acronyms

BWT	Burrows-Wheeler Transform
CC	Combinatorial Constraints
CSV	Comma-Separated Variables
DFT	Discrete Fourier Transform
DV	Developing Variation
FFT	Fast Fourier Transform
GMC	Global Morphological Constraints
Gv/Ga	Gödel-vector/Gödel-address
HSA	Hypothesis of Self-Similarity in Euclidean Axiomatics
ICMM	International Congress on Music and Mathematics (Puerto Vallarta 2014)
LMC	Local Morphological Constraints
MCT	Musical Contour Theory
NDFSA	Non-Deterministic Finite State Automata
OGMC	Optimal Global Morphological Constraints
PA	Partitional Analysis
PYL	Partitional Young Lattice
RTM	Rhythm in arrays notation (from RTM-notation to ENP-score-notation)
SAT	Self-referential Abstract Thought
TIP	Theory of Integer Partitions
TM	Tonal Music
UDP	User Datagram Protocol

Introduction

Emilio Lluis-Puebla

For those who read for the first time or inquire about *music and mathematics*, let me tell you that this field is both a recent area of study and also a very old one. At the beginning of history, there was a connection between numbers and music. Later, Pythagoras made a mathematical effort to say things about music with a certain foundation. The names Descartes, Galileo, Kepler, Leibniz, Euler, d'Alembert, Helmholtz, and some others are relevant here.

In the twentieth century, acoustics and its technology were very successful applying mathematics to music, as well as computer science and some other fields like linguistics. Later, the work of Clough in 1979, Lewin in 1982, and Mazzola in 1985 inspired both music-inclined mathematicians and mathematics-inclined musicians to continue working in mathematics and music.

A big trend in the last three decades in mathematics was to do not only applications but to do new mathematics in a variety of fields of knowledge, and the field of music has been no exception.

So, mathematical music theory is both a recent area of study and also a very old one. From Pythagoras until the 1980s, very little and not very sophisticated mathematics was employed in music. When sufficiently powerful mathematical machinery became available and talented mathematicians used it, modern mathematical music theory was born.

One of the main goals of mathematical music theory (I will state some of Guerino Mazzola's thoughts mainly from [1] and from personal conversations with him) was to develop a scientific framework for musicology. This framework had as its foundation, established scientific fields. It included a formal language for musical and musicological objects and relations. Music is fundamentally rooted within physical, psychological and semiotic realities. But the formal description of musical instances corresponds to mathematical formalism.

Mathematical music theory is based on category theory, algebraic topology, in particular, topos theory, module theory, group theory, homotopy theory, homology theory, algebraic geometry, just to name some areas, that is, on heavy mathematical machinery. Its purpose is to describe musical structures. The philosophy behind it is understanding the aspects of music that are susceptible to reason in the same way as physics does it for natural phenomena.

This theory is based in an appropriate language to manage the relevant concepts of the musical structures, in a group of postulates or theorems with regard to the musical structures subject to the defined conditions, and in the functionality for composition and analysis with or without a computer.

Mazzola also says that music is a central issue in human life, though it affects a different layer of reality than physics. The attempt to understand or to compose a major work of music is as important and difficult as the attempt to unify gravitation, electromagnetism, and weak and strong forces. For sure, the ambitions are comparable and hence the tools should be comparable too.

It is only in the last three decades that there is consistent work in mathematical music theory. Thus I will address this period of time in Mexico's history on this subject, since Gabriel Pareyon [2] summarises the time span before 1980. I will write about this in a personal way.

When I was 21 years old, in 1974, I was listening to the station Radio Universidad (University Radio Station), to a low, magnificent voice that was talking (in Spanish) about the application of finite group theory to the musical analysis of Bach's music, etc. This caught my attention and I went to see the owner of this voice. I located him in the old building of the Escuela Nacional de Música de la UNAM (UNAM Faculty of Music) and this young thin man kindly showed me a bunch of papers he had. I read them for half an hour or so and got the idea of what he was doing. This young man was Julio Estrada, a distinguished Mexican composer and musicologist.

Then I went to Canada to do a Ph.D. on algebraic K-theory. I was in love with pure mathematics like homological algebra, algebraic topology, algebraic geometry, homotopy theory, etc. Nobody could have ever told me that these marvellous pieces of pure mathematics were ever to appear more than thirty years later in the other field of my passion: music.

When I came back to México, in the early 1980s I wanted to do some work in mathematics and music, in particular to guide an undergraduate thesis for a student, but the angry face and terrible gesticulations of a colleague who was in charge of some high position at the department demoralised me. Does this sound familiar to anyone?

Some years later, in the 1990s, a lady from the mathematics undergraduate program at UNAM with a piano background, with great conviction, full of energy, appeared in my office, completely determined to do an undergraduate thesis in mathematics and music, particularly based on the ideas of Julio Estrada which turned into a book that he published in the 1980s [3]. I gave her more papers and books and she started to look for more bibliography. The librarian got some references of Guerino Mazzola. Particularly his book Gruppen und Categorien in der Musik, some articles by him, and others, including Chemiller's papers, plus some from the American School. This lady was Mariana Montiel. Now she is a full professor in the United States.

Mariana decided also to do a master's thesis on mathematical music theory, especially on denotator theory. I invited Guerino Mazzola to México for the first time in 1997 and we began a wonderful friendship.

In 2000, when I was President of the Sociedad Matemática Mexicana (Mathematical Society of Mexico), I dared to organise the First International Seminar on Mathematical Music Theory which took place simultaneously at the Facultad de Ciencias (Faculty of Sciences) and the Escuela Nacional de Música (School of Music) both from UNAM. Thomas Noll and Guerino Mazzola attended, among others.

Some days before the first international seminar, we had a previous special session on mathematical music theory at the annual Congreso Nacional de la Sociedad Matemática Mexicana in Saltillo which had an attendance of about 2000 persons, with great success. As a frame to both meetings we had concerts by Guerino Mazzola in Saltillo, Sala Carlos Chávez and at the Sala Xochipilli in Mexico City which turned into a delightful free jazz recording called Folia: The UNAM Concert with Guerino Mazzola playing Rachmaninoff's Corelli: La Folia theme as motive.

At both meetings, many mathematicians and musicians attended with surprise on their faces. The proceedings of the seminar were published by the Sociedad Matemática Mexicana Electronic Publications and lately were unified with the proceedings of the Second International Seminar which took place in Germany in 2001 and with the third one which took place in Switzerland in 2002 and was published by Epos Music of the University of Osnabruck in 2004 [4]. (I almost did not see this publication because I almost died. I was very ill for six months with an unknown disease which was later believed to be a viral meningitis, for which there was no cure!)

After not dying, six years later, in 2009, a student of mine, a young, impetuous and talented mathematician and musician, Octavio Agustin-Aquino, convinced me to organise the fourth seminar. It took place in Huatulco, Oaxaca, in 2010 as the Fourth International Seminar on Mathematical Music Theory [5]. By the way, Octavio became the first Ph.D. in mathematics graduated in Mexico at UNAM in mathematical music theory in 2011 with a thesis on microtonal counterpoint. He is now a full professor at the Universidad de la Cañada which belongs to the SUNEO in Oaxaca State, Mexico.

Finally, in November 2012 another very talented man (musicologist, also doing systematic musicology) which I admire the most because of his vast culture, ability, organisational capabilities, enormous memory and many other wits, contacted me in order to organise a sequel of the international seminars which turned out to be the International Congress on Music and Mathematics, 2014. This great man is Gabriel Pareyon.

Through the years there were also some more students who did some work with me but they did not continue in this field due to economic or vocational reasons. In 2013 and 2014, two of my students (Yemile Chávez and Santiago Rovira, both with music backgrounds) approached me like Mariana and Octavio before. They presented a lecture at ICMM 2014, and I hope they continue to work in this marvellous field.

Of course there are some other colleagues who have worked in mathematics and music in a rather isolated way, but now we had the opportunity to collect their efforts in this book, and made the connections to have a stronger unified community worldwide.

And well, what relationship does exist between music and mathematics? Or equivalently what connection or correspondence exists? We know, for example, that mathematical concepts were applied several years ago and recently (coming after all from nature or from man's abstract thought, etc.), just to mention four examples I use in my lectures [6]: to the entertainment with a game of dice in Mozart's creations; to aesthetics, as in Birkhoff's theory; to musical composition, for example by Bartók; and to create a precise language for musicology and music by Mazzola, among others. Certainly, there are many other music fields where mathematics contributes to our understanding, like in performance or analysis, etc.

For me, the most important relationship between mathematics and music is that both are "fine arts". They possess similar characteristics. They are related in the sense that mathematics provides a way to understand music, and musicology has a scientific basis in order to be considered a science, not a branch of common poetic literature.

I have worked since the 1970s on homotopy theory, cohomology theory, algebraic topology, homological algebra, among other fields of mathematics. As I wrote before, at the time these were considered pure mathematics. However, thirty years later, these wonderful pieces of mathematics came to be applied mathematics, and guess where? It turned out to be (as I wrote before) in my other passion: music! But not only as an application, you can do new mathematics as well!

Let me tell you an anecdote. In 2001, when I was president of the Sociedad Matemática Mexicana, during a visit to Rio de Janeiro I called a friend of mine, the president of the International Mathematical Union at that time, the Brazilian Jacob Palis. We agreed to meet at the famous Copacabana Palace where I was going to play Rachmaninoff's Second Piano Concerto as a soloist of the Rio de Janeiro Philharmonic Orchestra. He did not know I was a pianist. When he got there, he saw the president of the Sociedad Matemática Mexicana getting out on stage and sitting down to play the concerto. He was thrilled and invited me to dinner. We had a very long talk and having answered all his questions about me as a pianist and about mathematical music theory, he told me almost the same phrase that Guerino Mazzola got from Grothendieck: "the mathematics of the future!". So, in brief words, let me tell you that, for me, mathematics is one of the "fine arts", the purest of them, which has the gift of being the most precise of all sciences.

I was very honoured to meet all of the participants of ICMM in order to stimulate the interchange of visions, thoughts and points of view on this fascinating subject in a very friendly way. I am sure we all have profited from this interaction in such a wonderful place.

As you know, not only in Mexico, the funding for meetings is practically nonexistent. Many persons interested in coming could not join us because they did not have economic support from their universities. We thought we could obtain some funding for it, but once more, as in the Fourth International Seminar, we had to do it with our own personal budgets, energies and personal work and risk. We proudly can say that once more we have done it by ourselves!

Besides the small support (for such a big meeting) of very few institutions (see the acknowledgements in this book) we only had a small contribution from the Sociedad Matemática Mexicana to partially finance two of my own students which we, again, sincerely thank. The rest is exclusively ours and yours.

On Gabriel Pareyon's behalf (I recognise all his tremendous work on the organisation), the other organisers and myself, we thank all the participants of the International Congress on Music and Mathematics. We had a wonderful conference!

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Extended Counterpoint Symmetries and Continuous Counterpoint

Octavio A. Agustín-Aquino

Abstract A counterpoint theory for the whole continuum of the octave is obtained from Mazzola's model via extended counterpoint symmetries, and some of its properties are discussed.

1 Introduction

Mazzola's model for first species counterpoint is interesting because it predicts the rules of Fux's theory (in particular, the forbidden parallel fifths) reasonably well. It is also generalizable to microtonal equally tempered scales of even cardinality, and offers alternative understandings of consonance and dissonance distinct from the one explored extensively in Europe. In this paper we take some steps towards an effective *extension* of the whole model from a microtonal equally tempered scale into another, and not just of the mere consonances and dissonances, as it was done by the author in his doctoral dissertation [1].

First, we provide a definition of an *extended* counterpoint symmetry that preserves the characteristics of the counterpoint of one scale in the refined one. Then, we see that the progressive granulation of a specific example suggest an infinite counterpoint with a continuous polarity, different from the one that Mazzola himself proposed; a comparison of both alternatives calls for a deeper examination of the meaning of counterpoint extended to the full continuum of frequencies within the octave.

We must warn the reader that just a minimum exposition of Mazzola's counterpoint model is done, and hence we refer to his treatise *The Topos of Music* [3] (whose notation we use here) and an upcoming comprehensive reference [5] for further details.

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2 Some Definitions and Notations

Let R be a finite ring of cardinality $2k$. A subset S of R of such that $|S| = k$ is a *dichotomy*. It is often denoted by $(S/\complement S)$ to make the complement explicit. The group

$$\overrightarrow{GL}(R) = R \ltimes R^{\times} = \{e^u v : u \in R, v \in R^{\times}\}$$

is called the *affine group* of R, its members are the *affine symmetries*. It acts on R by

$$e^u v(x) = vx + u;$$

this action is extended to subsets in a pointwise manner. A dichotomy S is called *self-complementary* if there exists an affine symmetry p (its *quasipolarity*) such that $p(S) = \complement S$. A self-complementary dichotomy is *strong* if its quasipolarity p is unique, in which case p is called its *polarity*.

Of particular interest are the strong dichotomies of $\mathbb{Z}_{2k}$, since this ring models very well the equitempered $2k$-tone scales modulo octave and Mazzola discovered that the set of classical consonances is a strong dichotomy. For counterpoint, the self-complementary dichotomies of the dual numbers

$$\mathbb{Z}_{2k}[\epsilon] = \{a + \epsilon.b : a, b \in \mathbb{Z}_{2k}, \epsilon^2 = 0\}$$

are even more interesting, since they are used in Mazzola's counterpoint model as counterpoint intervals. More specifically, given a counterpoint interval $a + \epsilon.b$, a represents the cantus firmus, and b the interval between a and the discantus, and from every strong dichotomy (K/D) with polarity $p = e^u.v$ in $\mathbb{Z}_{2k}$ we can obtain the *induced interval dichotomy*

$$(K[\epsilon]/D[\epsilon]) = \{x + \epsilon.k : x \in \mathbb{Z}_{2k}, k \in K\}$$

in $\mathbb{Z}_{2k}[\epsilon]$. It is easily proved that, for every cantus firmus, there exists a quasipolarity $q_x[\epsilon]$ that leaves its *tangent space* $x + \epsilon.K$ invariant.

A symmetry $g \in \overrightarrow{GL}(\mathbb{Z}_{2k}[\epsilon])$ is a *counterpoint symmetry* of the consonant interval $\xi = x + \epsilon.k \in K[\epsilon]$ if

1. the interval ξ belongs to $g(D[\epsilon])$,
2. it commutes with the quasipolarity $q_x[\epsilon]$,
3. the set $g(K[\epsilon]) \cap K[\epsilon]$ is of maximal cardinality among those obtained with symmetries that satisfy the previous two conditions.

Given a counterpoint symmetry g for a consonant interval ξ, the members of the set $g(K[\epsilon]) \cap K[\epsilon]$ are its *admitted successors*; they represent the rules of counterpoint in Mazzola's model. It must also be noted that it can be proved that the admitted successors only need to be calculated for intervals of the form $0 + \epsilon.k$, and then suitably transposed for the remaining intervals.

3 Extending Counterpoint Symmetries

Let (X_n/Y_n) be a strong dichotomy in $\mathbb{Z}_n$ where

$$g_1 = e^{\epsilon.t_1}(u_1 + \epsilon.u_1 v_1) : \mathbb{Z}_n[\epsilon] \to \mathbb{Z}_n[\epsilon]$$

is a contrapuntal symmetry for the consonant interval $\epsilon.y \in X_n[\epsilon]$, with $p_n = e^{r_1} w_1$ the polarity of (X_n/Y_n). This means that if $s \in X_n$ and $p_n[\epsilon] = e^{\epsilon.r_1} w_1$ is the induced quasipolarity then

$$t_1 = y - u_1 p_n(s) \quad \text{and} \quad p_n[\epsilon](\epsilon.t_1) = g_1(\epsilon.r_1),$$

as it is proved in [3, p. 652]. If $a : X_n \hookrightarrow X_{an} : x \mapsto ax$ is an embedding of dichotomies, then

$$p_{an} \circ a = a \circ p_n$$

(where $p_{an} = e^{r_2} w_2$ is the polarity of (X_{an}/Y_{an})) and, evidently,

$$p_{an}[\epsilon] \circ a = a \circ p_n[\epsilon].$$

In particular, $ar_1 = r_2$.

Suppose there is a symmetry

$$g_2 = e^{\epsilon.t_2}(u_2 + \epsilon.u_2 v_2) : \mathbb{Z}_{an}[\epsilon] \to \mathbb{Z}_{an}[\epsilon]$$

such that $a \circ g_1 = g_2 \circ a$, then

$$t_2 = at_1 \quad \text{and} \quad au_2 = au_1.$$

From this we deduce

$$\begin{aligned} t_2 = at_1 &= ay - au_1 p_1(s) \\ &= ay - u_2 a p_n(s) \\ &= ay - u_2 p_{an}(as) \end{aligned}$$

where $as \in X_{an}$, and

$$\begin{aligned} p_{an}[\epsilon](\epsilon.t_2) &= p_{an}[\epsilon](\epsilon.at_1) \\ &= a p_n[\epsilon](\epsilon.t_1) = a g_1(\epsilon.r_1) \\ &= g_2(\epsilon.ar_1) = g_2(\epsilon.r_2). \end{aligned}$$

This means that g_2 is almost a contrapuntal symmetry for $\epsilon.ay$, except for the maximization of the intersection $g_2 X_{an}[\epsilon] \cap X_{an}[\epsilon]$. Now we can define a *extended counterpoint symmetry with respect the embedding a* as a symmetry $g_2 \in \overrightarrow{GL}(\mathbb{Z}_{an}[\epsilon])$ that satisfy

1. $a \circ g_1 = g_2 \circ a$ with g_1 a (extended or not) contrapuntal symmetry for $\epsilon.y$, and
2. $g_2 X_{an}[\epsilon] \cap X_{an}[\epsilon]$ has the maximum cardinality among the symmetries with the above property.

Note that extended counterpoint symmetries preserve the admitted successors of $\epsilon.y \in \mathbb{Z}_n[\epsilon]$, since otherwise the restriction $g_2|_{\mathbb{Z}_n[\epsilon]}$ of a extended counterpoint symmetry would be a symmetry such that the intersection $g_2|_{\mathbb{Z}_n[\epsilon]} X_n[\epsilon] \cap X_n[\epsilon]$ is bigger than the corresponding intersection for any counterpoint symmetry. This is a contradiction.

Remark 1 In particular, extended counterpoint symmetries always exist in the case of the embedding $2 : \mathbb{Z}_n \to \mathbb{Z}_{2n}$, because all the elements of $GL(\mathbb{Z}_n)$ are coprime with 2. Thus, for any $\epsilon.y \in \lim_{k\to\infty} X_{2^k \cdot n}[\epsilon]$, there exist a extended contrapuntal symmetry in the limit $\lim_{k\to\infty} \mathbb{Z}_{2^k \cdot n}[\epsilon]$ which is the limit of extended counterpoint symmetries.

Example 1 Let $X_6 = \{0, 2, 3\} \subseteq \mathbb{Z}_6$. The consonant interval $\epsilon.2 \in \mathbb{Z}_6[\epsilon]$ has $e^{\epsilon.3}$ $(1 + \epsilon.3)$ as its only counterpoint symmetry and 15 admitted successors. The extended counterpoint symmetries of $\epsilon.4 \in X_{12} = \{0, 1, 4, 5, 6, 9\} \subseteq \mathbb{Z}_{12}$ with respect to the embedding 2 are $e^{\epsilon.6}.(1 + \epsilon.6)$ and $e^{\epsilon.6}.(7 + \epsilon.6)$. The number of extended admitted successors is 48.

4 A More Detailed Example

In Example 4.11 of [1], it is shown that there exists a strong dichotomy in $\mathbb{Z}_{24}$ that can be extended progressively (via the embedding Lemma 4.5 of [1]) towards a dense dichotomy in S^1 with polarity $x \mapsto xe^{i\pi}$, which is the antipodal map. Analogously, the dichotomy

$$U_0 = \{0, 1, 3, \ldots, 7, 10\}$$

in $\mathbb{Z}_{16}$ can be completed in each step using the dichotomy

$$V_i = \{0, \ldots, |U_i| - 1\},$$

so we have the inductive definition

$$U_{i+1} = 2U_i \cup (2V_i + 1), \quad i \geq 1,$$

which is a strong dichotomy of $\mathbb{Z}_{2^{4+i}}$, in each case with polarity $e^{2^{3+i}}$. Note that the injective limit of the U_i in S^1 is dense in one hemisphere.

The standard counterpoint symmetries for U_0 and successively extended counterpoint symmetries for $\mathbb{Z}_{512}$ are listed in Table 1. With "successively extended" we mean that they are those who commute with the extended counterpoint symmetries of $\mathbb{Z}_{256}$, which in turn commute with those of $\mathbb{Z}_{128}$, and so on down to $\mathbb{Z}_{16}$. In most cases the linear part is -1, and in fact it is remarkable that all of them have no dual component.

5 A Possible Continuous Counterpoint

The previous calculations suggest the following constructions that enable a continuous and compositionally useful counterpoint. First, we consider the space $S^1 \subseteq \mathbb{C}$ (which represents the continuum of intervals modulo octave), with the action of the group $G = \mathbb{R}/\mathbb{Z} \ltimes \mathbb{Z}_2$ given by

Table 1 A set of consonances in $\mathbb{Z}_{16}$, their respective counterpoint symmetries and number of admitted successors, and their extended counterpoint symmetries when embedded in $\mathbb{Z}_{512}$, with the corresponding number of extended admitted successors

Interval	Symmetries for $\mathbb{Z}_{16}$	$\lvert gX[\epsilon] \cap X[\epsilon]\rvert$	Extended symmetries for $\mathbb{Z}_{512}$	$\lvert gX[\epsilon] \cap X[\epsilon]\rvert$
0	$e^{\epsilon 5}3$	96		
	$e^{\epsilon 6}13$			
	$e^{\epsilon 11}15$		$e^{\epsilon 352}511$	82432
1	$e^{\epsilon 10}15$	112	$e^{\epsilon 320}511$	98816
3	$e^{\epsilon 2}5$	96		
	$e^{\epsilon 9}11$			
	$e^{\epsilon 11}15$		$e^{\epsilon 352}511$	82432
4	7	112	7	75264
			439	
5	$e^{\epsilon 1}3$	96		
	$e^{\epsilon 6}13$			
	$e^{\epsilon 7}15$		$e^{\epsilon 244}511$	124416
6	$e^{\epsilon 3}13$	112	$e^{\epsilon 96}205$	76800
7	$e^{\epsilon 1}5$	112	$e^{\epsilon 16}5$	76800
10	$e^{\epsilon 2}5$	96		
	$e^{\epsilon 5}11$			
	$e^{\epsilon 7}15$		$e^{\epsilon 244}511$	124416

$$e^t v(x) = \begin{cases} x \exp(2\pi i t), & v = 1, \\ \overline{x} \exp(2\pi i t), & v = -1. \end{cases}$$

We define the set of consonances (K/D) as the image of $[0, \frac{1}{2})$ under the map $\phi : [0, 1] \mapsto S^1 : t \mapsto e^{2i\pi t}$, which musically means that we consider as consonant any interval greater or equal than the unison but smaller than the tritone (within an octave). Apart from the identity, no element of G leaves (K/D) invariant, thus it is strong and its polarity is $e^{\frac{1}{2}}$.

Now, for counterpoint, we consider the torus $T = S^1 \times S^1$, with the first component for the cantus firmus and the second for the discantus interval. Let G act on T in the following manner:

$$e^t v(x, y) = (vx, e^t vy);$$

this action is suggested by the fact that all the linear parts of the affine symmetries of counterpoint intervals have no dual component.

Thus the set of consonant intervals is $(K[\epsilon]/D[\epsilon]) = (S^1 \times K/S^1 \times D)$, the self-complementary function for any $\xi \in T$ which fixes its tangent space is $e^{1/2}1$, and it commutes with any element of G'. Also $\xi = (0, k) \in g(D[\epsilon])$ for a $g \in G'$ if and only if

$$g = e^t 1, \quad t \in (k, k + 1/2] \quad \text{or} \quad g = e^t(-1), \quad t \in [k - 1/2, k).$$

And here comes a delicate point. If we wish to preserve the idea of cardinality maximization, it would be reasonable to ask the set of infinite admitted successors to attain certain maximum. A possibility is to gauge these sets in terms of the standard measure in T since, for instance, the affine morphisms

$$g = \begin{cases} e^{k-1/2}(-1), & k \in \phi([0, 1/4]), \\ e^{k-1/2}1, & k \in \phi([1/4, 1/2]), \end{cases}$$

maximize the measure of the intersection $(gX[\epsilon]) \cap X[\epsilon]$. The musical meaning of this alternative is that the admitted successors of consonant intervals below the minor third are all the consonant intervals above it, and vice versa. The minor third is special, because it has any consonant interval as an admitted successor.

But, in terms of the new perspective of homology introduced by Mazzola in [4], we observe first that T is homeomorphic to T itself with respect to the Kuratowski closure operator induced by the quasipolarity $e^{1/2}1$. This is so because, for in each section $x \times S^1$, the self-complementary function is the antipodal morphism, thus each $x \times S^1$ is homeomorphic to the projective line, which in turn is homeomorphic to $x \times S^1$ itself [2, p. 58]. Furthermore, any $g \in G'$ which leaves ξ out of $g(X[\epsilon])$ is such that $(g(X[\epsilon])) \cap X[\epsilon]$ is homotopically equivalent to S^1, except when such