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The Local Langlands Conjecture for GL(2)

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For Elisabeth and Lesley

Foreword

This book gives a complete and self-contained proof of Langlands' conjecture concerning the representations of GL(2) of a non-Archimedean local field. It has been written to be accessible to a doctoral student with a standard grounding in pure mathematics and some extra facility with local fields and representations of finite groups. It had its origins in a lecture course given by the authors at the first Beijing-Zhejiang International Summer School on p-adic methods, held at Zhejiang University Hangzhou in 2004. We hope this is found a fitting response to the efforts of the organizers and the enthusiastic contribution of the student participants.

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Contents

Introduction

We work with a non-Archimedean local field F which, we always assume, has finite residue field of characteristic p. Thus F is either a finite extension of the field $\mathbb{Q}_p$ of p-adic numbers or a field $\mathbb{F}_{p^r}((t))$ of formal Laurent series, in one variable, over a finite field. The arithmetic of F is encapsulated in the *Weil group* $\mathcal{W}_F$ of F: this is a topological group, closely related to the Galois group of a separable algebraic closure of F, but with rather more sensitive properties. One investigates the arithmetic via the study of continuous (in the appropriate sense) representations of $\mathcal{W}_F$ over various algebraically closed fields of characteristic zero, such as the complex field $\mathbb{C}$ or the algebraic closure $\overline{\mathbb{Q}}_\ell$ of an ℓ-adic number field.

Sticking to the complex case, the one-dimensional representations of $\mathcal{W}_F$ are the same as the characters (i.e., continuous homomorphisms) $F^\times \to \mathbb{C}^\times$: this is the essence of *local class field theory.* The n-dimensional analogue of a character of $F^\times = \mathrm{GL}_1(F)$ is an irreducible smooth representation of the group $\mathrm{GL}_n(F)$ of invertible $n \times n$ matrices over F. As a specific instance of a wide speculative programme, Langlands [55] proposed, in a precise conjecture, that such representations should parametrize the n-dimensional representations of $\mathcal{W}_F$ in a manner generalizing local class field theory and compatible with parallel global considerations.

The excitement provoked by the local Langlands conjecture, as it came to be known, stimulated a period of intense and widespread activity, reflected in the pages of [8]. The first case, where $n = 2$ and F has characteristic zero, was started in Jacquet-Langlands [46]; many hands contributed but Kutzko, bringing two new ideas to the subject, completed the proof in [52], [53]. Subsequently, the conjecture has been proved in all dimensions, first in positive characteristic by Laumon, Rapoport and Stuhler [58], then in characteristic zero by Harris and Taylor [38], also by Henniart [43] on the basis of an earlier paper of Harris [37].

Throughout the period of this development, the subject has largely remained confined to the research literature. Our aim in this book is to provide a navigable route into the area with a complete and self-contained account of the case $n = 2$, in a tolerable number of pages, relying only on material readily available in standard courses and texts. Apart from a couple of unavoidable caveats concerning Chapter VII, we assume only the standard representation theory for finite groups, the beginnings of the theory of local fields and some very basic notions from topology.

In consequence, our methods are entirely local and elementary. Apart from Chapter I (which could equally serve as the start of a treatise on the representation theory of p-adic reductive groups) and some introductory material in Chapter VII, we eschew all generality. Whenever possible, we exploit special features of GL(2) to abbreviate or simplify the arguments.

The desire to be both compact and complete removes the option of appealing to results derived from harmonic analysis on adèle groups ("base change" [57], [1]) which originally played a determining rôle. This particular constraint has forced us to give the first proof of the conjecture that can claim to be completely local in method.

There is an associated loss, however. The local Langlands Conjecture is just a specific instance of a wide programme, encompassing local and global issues and all connected reductive algebraic groups in one mighty sweep. Beyond the minimal gesture of Chapter XIII, we can give the reader no idea of this. Nor have we mentioned any of the geometric methods currently necessary to prove results in higher dimensions. Fortunately, the published literature contains many fine surveys, from Gelbart's book [32], which still conveys the breadth and excitement of the ideas, to the new directions described in [4].

The approach we take is guided by [46] and [50–53], but we have rearranged matters considerably. We have separated the classification of representations from the functional equation. We have imported ideas of Bernstein and Zelevinsky into the discussion of non-cuspidal representations. While the treatment of cuspidal representations is essentially that of Kutzko, it is heavily informed by hindsight. We have given precedence to the Godement-Jacquet version of the functional equation and so had to treat the Converse Theorem in a novel manner, owing something to ideas of Gérardin and Li. There is also some degree of novelty in our treatment of the Kirillov model and the relation between the functional equation it gives and that of Godement and Jacquet. We have given a quick and explicit proof of the existence of the Langlands correspondence, in the case $p \neq 2$, at an early stage.

The case $p = 2$ has many pages to itself. The method is essentially that of Kutzko, but we have had to bring a new idea to the closing pages (the treatment of the so-called octahedral representations) to avoid an appeal to base change. We regard this case as being particularly important. It remains the one instance of the local conjecture in which the detail is sufficiently complex to be interesting, yet sufficiently visible to illuminate the miracle that is the Langlands correspondence. Even after 25 years, it stands as a sturdy corrective to over-optimistic attitudes to more general problems.

As light relief, we have broadened the picture with some discussion of ℓ-adic representations, since these provide a forum in which the correspondence finds much of its application.

The final Chapter XIII stands outside the main sequence. There, D is the quaternion division algebra over F. The irreducible representations of $D^\times = \mathrm{GL}_1(D)$ can be classified by a method parallel to that used for $\mathrm{GL}_2(F)$. The Jacquet-Langlands correspondence provides a canonical connection between the representation theories of $D^\times$ and $\mathrm{GL}_2(F)$. We include it as an indication of further dimensions in the subject. Given the experience of $\mathrm{GL}_2(F)$, it is a fairly straightforward matter which we have left as a sequence of exercises.

Acknowledgement. The final draft was read by Corinne Blondel, whose acute comments led us to remove a large number of minor errors and obscurities, along with a couple of more significant lapses. It is a pleasure to record our debt to her.

Notation

We list some standard notations which we use repeatedly, without always recalling their meaning.

F = *a non-Archimedean local field;*
$\mathfrak{o}$ = *the discrete valuation ring in* F*;*
$\mathfrak{p}$ = *the maximal ideal of* $\mathfrak{o}$*;*
$\boldsymbol{k}$ = $\mathfrak{o}/\mathfrak{p}$; p = *the characteristic of* $\boldsymbol{k}$*;* $q = |\boldsymbol{k}|$;
U_F = *the group of units of* $\mathfrak{o}$*;* $U_F^n = 1+\mathfrak{p}^n$, $n \geqslant 1$.

(Thus the characteristic of F is 0 or p: we never need to impose any further restriction.) In addition, $\upsilon_F : F^\times \to \mathbb{Z}$ is the normalized (surjective) additive valuation and $\|x\| = q^{-\upsilon_F(x)}$. We denote by $\boldsymbol{\mu}_F$ the group of roots of unity in F of order prime to p.

If E/F is a finite field extension, we use the analogous notations $\mathfrak{o}_E$, $\mathfrak{p}_E$, etc. The norm map $E^\times \to F^\times$ is denoted $\mathrm{N}_{E/F}$, and the trace $E \to F$ is $\mathrm{Tr}_{E/F}$. The ramification index and the residue class degree are $e(E|F)$, $f(E|F)$ respectively. The discriminant is $\mathfrak{d}_{E/F} = \mathfrak{p}^{d+1}$, $d = d(E|F)$.

The symbol tr is reserved for the trace of an endomorphism, such as a matrix or a group representation, and det is invariably the determinant.

If R is a ring with 1, $R^\times$ is its group of units and $\mathrm{M}_n(R)$ is the ring of $n\times n$ matrices over R. When R is commutative, $\mathrm{GL}_n(R)$ (resp. $\mathrm{SL}_n(R)$) is the group of $n \times n$ matrices over R which are invertible (resp. of determinant 1). We use the notation B, T, N, Z for the subgroups of GL_2 of matrices of the form

$$\begin{pmatrix} * & * \\ 0 & * \end{pmatrix}, \quad \begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix}, \quad \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}$$

respectively. Unless otherwise specified, $A = \mathrm{M}_2(F)$ and $G = \mathrm{GL}_2(F)$.

Notes for the reader

Prerequisites. We assume the beginnings of the representation theory of finite groups, including Mackey theory: the first 11 sections of [77] cover it all, bar a couple of results requiring reference to [26]. Of non-Archimedean local fields, we need general structure theory as far as the discriminant and structure of tame extensions, plus behaviour of the norm in tame or quadratic extensions. Practically everything can be found in [30] or the first two parts of [76], while [87] is the source of many of the ideas here. From topology and measure theory, beyond the most elementary concepts, we cover practically everything we need.

All this material is commonly available in many books: we mention only personal favourites.

From Chapter VII onwards, we rely on local class field theory. No detail is involved, so we have been able to take an axiomatic approach. The reader might consult the compact [68] or [74], [76]. More serious is the treatment in §30 of the existence of the Langlands-Deligne local constant. This depends on an interplay between local and global fields using some deep (but classical) theorems. The reader could again take an axiomatic approach. We have included a brief account which is complete modulo the classical background. (The requisite material is in [68] or [54].)

Navigation. Sections are numbered consecutively throughout the book. Each section is divided into (usually) short paragraphs, numbered in the form y.z. A reference y.z Proposition means the (only) proposition in paragraph y.z.

Chapter I stands alone, and could serve as an introduction to much wider areas. Chapter II is elementary, and could be read first. Parts of Chapters VII and X can be read independently. Chapter XIII could be read directly after Chapter VI. Otherwise, the logical dependence is linear and fairly rigid.

Principal series (or non-cuspidal) representations form a distinct subtheme. At a first reading, this could be edited out or pursued exclusively, according to taste. (For a different approach, emphasizing non-cuspidal representations and their importance for L-functions, see Bump's book [10].) Another "short course" option would be to stop at the end of Chapter VIII, by which stage the argument is complete for all but dyadic fields F.

Exercises. A few exercises are scattered through the text. These are intended to illuminate, entertain, or to indicate directions we do not follow. Only the simpler ones ever make a serious contribution to the main argument.

Notes. We have appended brief notes or comments to some chapters, to indicate further reading or wider perspectives. They tend to pre-suppose greater experience than the main text.

History. We have written an account of the subject, not its history: that would be a separate project of comparable scope. We have made no attempt at a complete bibliography. We have cited sources of major importance, and those we have found helpful in the preparation of this volume. We have also mentioned a number of recent works, along with older ones that, in our opinion, remain valuable to one learning the subject.

1

Smooth Representations

This chapter is introductory and foundational in nature. We define a class of topological groups, the *locally profinite groups,* and study their *smooth* representations on complex vector spaces. These representations are often infinite-dimensional, but smoothness imposes a drastic continuity condition. Nontheless, this class of objects is quite wide: it includes, for example, all representations of discrete groups.

We start by recalling some standard facts. We then develop the elementary aspects of smooth representation theory, very much guided by the ordinary representation theory of finite groups. We occasionally turn to non-Archimedean local fields as a source of examples. The topic of Haar measure and integration on topological groups necessarily enters the picture. Since we have only to deal with locally profinite groups, this is a straightforward matter of which we give just as much as we need.

While we will ultimately be concerned only with non-Archimedean local fields F and associated groups like $\mathrm{GL}_2(F)$, there is nothing to be gained from specialization at this stage. Looking beyond the confines of the present book, there is much to be lost. We therefore work, throughout this chapter, in quite extreme generality.

1. Locally Profinite Groups

In this section, we introduce and briefly discuss the notion of a locally profinite group. We concentrate on showing how this framework accommodates the non-Archimedean local fields and some associated groups and rings. We give very few proofs in the first four paragraphs: it is more a case of gathering together the pre-requisite threads.

We conclude the section with a couple of paragraphs about various characters associated with a non-Archimedean local field F. We make unceasing use of this material in the later chapters. More immediately, it gives us some examples to illuminate the general theory of the following sections.

1.1.

Definition. *A* locally profinite group *is a topological group G such that every open neighbourhood of the identity in G contains a compact open subgroup of G.*

For example, any discrete group is locally profinite. A closed subgroup of a locally profinite group is locally profinite. The quotient of a locally profinite group by a closed normal subgroup is locally profinite.

A locally profinite group is locally compact. If it is compact, it is *profinite* in the usual sense, that is, the limit of an inverse system of finite discrete groups. In fact, if G is a compact locally profinite group, it is not hard to show directly that the obvious map

$$G \longrightarrow \varprojlim G/K$$

is a topological isomorphism, where K ranges over the open normal subgroups of G.

In general, any open neighbourhood of 1 in a locally profinite group G contains a compact open subgroup K of G. As we have just seen, K is profinite: the terminology is therefore apt.

Remark. A locally profinite group is locally compact and totally disconnected. In the converse direction, it is known that a compact, totally disconnected topological group is profinite. Likewise, a locally compact, totally disconnected group is locally profinite, but we shall make no use of that fact.

1.2. Let F be a non-Archimedean local field. Thus F is the field of fractions of a discrete valuation ring $\mathfrak{o}$. Let $\mathfrak{p}$ be the maximal ideal of $\mathfrak{o}$ and $\boldsymbol{k} = \mathfrak{o}/\mathfrak{p}$ the residue class field. We will always assume that $\boldsymbol{k}$ is *finite,* and we will generally denote the cardinality $|\boldsymbol{k}|$ by q.

Let ϖ be a prime element of F, that is, an element such that $\varpi\mathfrak{o} = \mathfrak{p}$. Every element $x \in F^\times$ can be written uniquely as $x = u\varpi^n$, for some unit $u \in \mathfrak{o}^\times = U_F$, and some $n \in \mathbb{Z}$. (We use the notation $n = \upsilon_F(x)$.) The field F carries an absolute value

$$\|x\| = q^{-n} = q^{-\upsilon_F(x)}, \quad \|0\| = 0,$$

giving a metric on F, relative to which F is complete. In the metric space topology, F is a topological field. The fractional ideals

$$\mathfrak{p}^n = \varpi^n\mathfrak{o} = \{x \in F : \|x\| \leqslant q^{-n}\}, \quad n \in \mathbb{Z},$$

are open subgroups of F and give a fundamental system of open neighbourhoods of 0 in F.

Combining the definition of the topology with the completeness property, one sees that the canonical map

$$\mathfrak{o} \longrightarrow \varprojlim_{n\geqslant 1} \mathfrak{o}/\mathfrak{p}^n$$

is a topological isomorphism. Since $\boldsymbol{k}$ is finite, each group $\mathfrak{o}/\mathfrak{p}^n$ is finite, and the limit is compact. Each fractional ideal $\mathfrak{p}^n$, $n \in \mathbb{Z}$, is isomorphic to $\mathfrak{o}$ and so is compact. We conclude:

Proposition. *The additive group F is locally profinite, and F is the union of its compact open subgroups.*

1.3. The multiplicative group $F^\times$ is likewise a locally profinite group: the congruence unit groups $U_F^n = 1+\mathfrak{p}^n$, $n \geqslant 1$, are compact open, and give a fundamental system of open neighbourhoods of 1 in $F^\times$.

1.4. Let $n \geqslant 1$ be an integer. The vector space $F^n = F \times \cdots \times F$ carries the product topology, relative to which it is a locally profinite group. As a special case, the matrix ring $\mathrm{M}_n(F)$ is a locally profinite group under addition, in which multiplication of matrices is continuous.

The group $G = \mathrm{GL}_n(F)$ is an open subset of $\mathrm{M}_n(F)$; inversion of matrices is continuous, so G is a topological group. The subgroups

$$K = \mathrm{GL}_n(\mathfrak{o}), \qquad K_j = 1 + \mathfrak{p}^j\mathrm{M}_n(\mathfrak{o}), \quad j \geqslant 1,$$

are compact open, and give a fundamental system of open neighbourhoods of 1 in G. Thus $G = \mathrm{GL}_n(F)$ is a locally profinite group.

More generally, let V be an F-vector space of finite dimension n. The choice of a basis gives an isomorphism $V \cong F^n$, which we use to impose a topology on V. This topology is independent of the choice of basis. The remarks above apply equally to the algebra $\mathrm{End}_F(V)$ and the group $\mathrm{Aut}_F(V)$.

1.5. Let V be an F-vector space of finite dimension n. An $\mathfrak{o}$-*lattice* in V is a finitely generated $\mathfrak{o}$-submodule L of V such that the F-linear span FL of L is V.

Proposition. *Let L be an $\mathfrak{o}$-lattice in V. There is an F-basis $\{x_1, x_2, \ldots, x_n\}$ of V such that $L = \sum_{i=1}^n \mathfrak{o}x_i$.*

Proof. By definition, L has a finite $\mathfrak{o}$-generating set. Choose a minimal such set $\{y_1, y_2, \ldots, y_m\}$: we show that this is an F-basis of V. It certainly spans V. Suppose it is linearly dependent over F:

$$\sum_{1 \leqslant i \leqslant m} a_i y_i = 0,$$

with $a_i \in F$, not all zero. We can multiply through by an element of $F^\times$ and assume that all $a_i \in \mathfrak{o}$ and that at least one of them, a_j say, is a unit of $\mathfrak{o}$. Thus y_j is an $\mathfrak{o}$-linear combination of the other y_i, contrary to the minimality hypothesis. □

In particular, an $\mathfrak{o}$-lattice L is a compact open subgroup of V. The $\mathfrak{o}$-lattices in V give a fundamental system of open neighbourhoods of 0 in V.

More generally, a *lattice* in V is a compact open subgroup of V. Here we have:

Lemma. *Let L be a subgroup of V; then L is a lattice in V if and only if there exist $\mathfrak{o}$-lattices L_1, L_2 in V such that $L_1 \subset L \subset L_2$.*

Proof. Suppose $L_1 \subset L \subset L_2$, where the L_i are $\mathfrak{o}$-lattices. Since L contains L_1, it is open and hence closed. Since L is contained in L_2, it is compact.

Conversely, if L is a lattice in V, it must contain an $\mathfrak{o}$-lattice (since L is an open neighbourhood of 0), and so $FL = V$. In the opposite direction, we choose a basis $\{x_1, \ldots, x_n\}$ of V. The image of L under the obvious projection $V \to Fx_i$ is a compact open subgroup of Fx_i. It is therefore contained in a group of the form $\mathfrak{a}_i x_i$, for some fractional ideal $\mathfrak{a}_i = \mathfrak{p}^{a_i}$ of $\mathfrak{o}$. Thus $L \subset \mathfrak{o}L \subset \sum_i \mathfrak{a}_i x_i$, and this is an $\mathfrak{o}$-lattice. □

1.6. Let G be a locally profinite group.

Proposition. *Let $\psi : G \to \mathbb{C}^\times$ be a group homomorphism. The following are equivalent:*

(1) *ψ is continuous;*
(2) *the kernel of ψ is open.*

If ψ satisfies these conditions and G is the union of its compact open subgroups, then the image of ψ is contained in the unit circle $|z| = 1$ in $\mathbb{C}$.

Proof. Certainly (2) $\Rightarrow$ (1). Conversely, let $\mathcal{N}$ be an open neighbourhood of 1 in $\mathbb{C}$. Thus $\psi^{-1}(\mathcal{N})$ is open and contains a compact open subgroup K of G. However, if $\mathcal{N}$ is chosen sufficiently small, it contains no non-trivial subgroup of $\mathbb{C}^\times$ and so $K \subset \operatorname{Ker}\psi$.

The unit circle S^1 is the unique maximal compact subgroup of $\mathbb{C}^\times$. If K is a compact subgroup of G, then $\psi(K)$ is compact, and so it is contained in S^1. The final assertion follows. $\square$

We define a *character* of a locally profinite group G to be a continuous homomorphism $G \to \mathbb{C}^\times$. We usually write 1_G, or even just 1, for the trivial (constant) character of G.

We call a character of G *unitary* if its image is contained in the unit circle.

1.7. We will later make frequent use of another property of the local field F.

The set of characters of F is a group under multiplication; we denote it $\widehat{F}$. Since F is the union of its compact open subgroups $\mathfrak{p}^n$, $n \in \mathbb{Z}$, all characters of F are unitary (1.6 Proposition).

If ψ is a character and $\psi \neq 1$, there is a least integer d such that $\mathfrak{p}^d \subset \operatorname{Ker}\psi$.

Definition. *Let $\psi \in \widehat{F}$, $\psi \neq 1$. The* level *of ψ is the least integer d such that $\mathfrak{p}^d \subset \operatorname{Ker}\psi$.*

If we fix d, the set of characters of F of level $\leqslant d$ is the subgroup of $\psi \in \widehat{F}$ such that $\psi \mid \mathfrak{p}^d = 1$.

Proposition *(Additive duality).* *Let $\psi \in \widehat{F}$, $\psi \neq 1$, have level d.*

(1) *Let $a \in F$. The map $a\psi : x \mapsto \psi(ax)$ is a character of F. If $a \neq 0$, the character $a\psi$ has level $d - \upsilon_F(a)$.*
(2) *The map $a \mapsto a\psi$ is a group isomorphism $F \cong \widehat{F}$.*

Proof. Part (1) is immediate, and $a \mapsto a\psi$ is an injective group homomorphism $F \to \widehat{F}$.

Let $\theta \in \widehat{F}$, $\theta \neq 1$, and let l be the level of θ. Let ϖ be a prime element of F, and $u \in U_F$. The character $u\varpi^{d-l}\psi$ has level l, and so agrees with θ on $\mathfrak{p}^l$. The characters $u\varpi^{d-l}\psi$, $u'\varpi^{d-l}\psi$, $u, u' \in U_F$, agree on $\mathfrak{p}^{l-1}$ if and only if $u \equiv u' \pmod{\mathfrak{p}}$. The group $\mathfrak{p}^{l-1}$ has $q-1$ non-trivial characters which are trivial on $\mathfrak{p}^l$. As u ranges over U_F/U_F^1, the $q-1$ characters $u\varpi^{d-l}\psi \mid \mathfrak{p}^{l-1}$ are distinct, non-trivial, but trivial on $\mathfrak{p}^l$. Therefore one of them, say $u_1\varpi^{d-l}\psi \mid \mathfrak{p}^{l-1}$, equals $\theta \mid \mathfrak{p}^{l-1}$.

Iterating this procedure, we find a sequence of elements $u_n \in U_F$ such that $u_n\varpi^{d-l}\psi$ agrees with θ on $\mathfrak{p}^{l-n}$ and $u_{n+1} \equiv u_n \pmod{\mathfrak{p}^n}$. The Cauchy sequence $\{u_n\}$ converges to some $u \in U_F$ and we have $\theta = u\varpi^{d-l}\psi$. $\square$

Exercise. *Let L be a lattice in F and let χ be a character of L (in the sense of* 1.6*). Show there exists a character ψ of F such that $\psi \mid L = \chi$.*

1.8. We turn to the multiplicative group $F^\times$. Let χ be a character of $F^\times$. By 1.6 Proposition, χ is trivial on U_F^m, for some $m \geqslant 0$.

Definition. *Let χ be a non-trivial character of $F^\times$. The* level *of χ is defined to be the least integer $n \geqslant 0$ such that χ is trivial on U_F^{n+1}.*

We use the same terminology for characters of open subgroups of $F^\times$.

Observe that a character of $F^\times$ need not be unitary: for example, the map $x \mapsto \|x\|$ is a character. Note also that, in a related contrast to the additive case, $F^\times$ has a *unique maximal compact subgroup,* namely U_F.

The structure of the group of characters of $F^\times$ is more subtle than that of $\widehat{F}$. However, we shall make frequent use of a partial description in additive terms. Let m, n be integers, $1 \leqslant m < n \leqslant 2m$. The map $x \mapsto 1{+}x$ gives an isomorphism $\mathfrak{p}^m/\mathfrak{p}^n \cong U_F^m/U_F^n$. This gives an isomorphism of character groups $(\mathfrak{p}^m/\mathfrak{p}^n)\widehat{\ } \cong (U_F^m/U_F^n)\widehat{\ }$, and we can use 1.7 to describe the group $(\mathfrak{p}^m/\mathfrak{p}^n)\widehat{\ }$.

For this purpose, it is convenient to fix a character $\psi_F \in \widehat{F}$ *of level* 1. For $a \in F$, we define a function

$$\begin{aligned} \psi_{F,a} &: F \longrightarrow \mathbb{C}^\times, \\ \psi_{F,a}(x) &= \psi_F(a(x{-}1)). \end{aligned} \tag{1.8.1}$$

Proposition 1.7 then yields:

Proposition. *Let $\psi \in \widehat{F}$ have level* 1*. Let m, n be integers, $0 \leqslant m < n \leqslant 2m{+}1$. The map $a \mapsto \psi_{F,a} \mid U_F^{m+1}$ induces an isomorphism*

$$\mathfrak{p}^{-n}/\mathfrak{p}^{-m} \xrightarrow{\ \approx\ } (U_F^{m+1}/U_F^{n+1})\widehat{\ }.$$

Observe that, viewed as a character of U_F^{m+1}, the function $\psi_{F,a}$ has level $-\upsilon_F(a)$. Also, the condition relating m and n can be re-formulated as $\left[\frac{n}{2}\right] \leqslant m < n$, where $x \mapsto [x]$ denotes the greatest integer function.

Terminology. We will use analogues of the notion of *level,* as defined in this paragraph, in many contexts where we study representations of groups with a filtration indexed by the non-negative integers. As we shall see, it is very convenient. From this more general viewpoint, the definition in 1.7 for characters of F appears anomalous. This version is forced on us by a variety of historical conventions: the only point on which these agree is that a character of F of level zero must be trivial on $\mathfrak{o}$ but not on $\mathfrak{p}^{-1}$. This is so firmly established that it would be confusing to change it now.

2. Smooth Representations of Locally Profinite Groups

In this section, we introduce the notion of a *smooth* representation of a locally profinite group G. We develop the basic theory, along lines familiar from the ordinary representation theory of finite groups. New phenomena do arise, but the general outline is very similar to that of the standard theory.

2.1. Let G be a locally profinite group, and let (π, V) be a representation of G. Thus V is a complex vector space and π is a group homomorphism $G \to \mathrm{Aut}_{\mathbb{C}}(V)$. The representation (π, V) is called *smooth* if, for every $v \in V$, there is a compact open subgroup K of G (depending on v) such that $\pi(x)v = v$, for all $x \in K$. Equivalently, if V^K denotes the space of $\pi(K)$-fixed vectors in V, then

$$V = \bigcup_K V^K,$$

where K ranges over the compact open subgroups of G.

In practice, we will usually have to deal with representations of infinite dimension.

A smooth representation (π, V) is called *admissible* if the space V^K is finite-dimensional, for each compact open subgroup K of G.

Let (π, V) be a smooth representation of G; then any G-stable subspace of G provides a further smooth representation of G. Likewise, if U is a G-subspace of V, the natural representation of G on the quotient V/U is smooth. One says that (π, V) is *irreducible* if $V \neq 0$ and V has no G-stable subspace U, $0 \neq U \neq V$.

For smooth representations (π_i, V_i) of G, the set $\mathrm{Hom}_G(\pi_1, \pi_2)$ is just the space of linear maps $f : V_1 \to V_2$ commuting with the G-actions:

$$f \circ \pi_1(g) = \pi_2(g) \circ f, \quad g \in G. \tag{2.1.1}$$

With this definition, the class of smooth representations of G forms a category $\mathrm{Rep}(G)$. We remark that the category $\mathrm{Rep}(G)$ is *abelian.*

We say that two smooth representations (π_1, V_1), (π_2, V_2) of G are *isomorphic,* or *equivalent,* if there exists a $\mathbb{C}$-isomorphism $f : V_1 \to V_2$ satisfying (2.1.1).

Example 1. A character χ of G (1.6) can be viewed as a representation $\chi : G \to \mathbb{C}^\times = \mathrm{Aut}_{\mathbb{C}}(\mathbb{C})$. The representation $(\chi, \mathbb{C})$ is smooth. A one-dimensional representation of G is smooth if and only if it is equivalent to a representation defined by a character of G. Indeed, the set of isomorphism classes of one-dimensional smooth representations of G is in canonical bijection with the group of characters of G.

Example 2. Suppose that G is *compact,* hence profinite. Let (π, V) be an irreducible smooth representation of G. The space V is then *finite-dimensional.* For, if $v \in V$, $v \neq 0$, then $v \in V^K$, for an open subgroup K of G. The index $(G:K)$ is necessarily finite, and the set $\{\pi(g)v : g \in G/K\}$ spans V. Further, if $K' = \bigcap_{g \in G/K} gKg^{-1}$, then K' is an open normal subgroup of G of finite index, acting trivially on V. Thus V is effectively an irreducible representation of the finite discrete group G/K'.

In this more general context, one can still define the group ring $\mathbb{C}[G]$ as the algebra of finite formal linear combinations of elements of G. A smooth representation V of G is then a $\mathbb{C}[G]$-module. However, an arbitrary $\mathbb{C}[G]$-module need not provide a smooth representation of G, and so the group ring is not an effective tool for analyzing smooth representations. For this purpose, one has to replace $\mathbb{C}[G]$ by a different algebra. We discuss this in §4 below.

2.2. We recall a standard concept in the present context.

Proposition. *Let G be a locally profinite group, and let (π, V) be a smooth representation of G. The following conditions are equivalent:*

(1) *V is the sum of its irreducible G-subspaces;*
(2) *V is the direct sum of a family of irreducible G-subspaces;*
(3) *any G-subspace of V has a G-complement in V.*

Proof. We start with the implication (1) ⇒ (2). We take a family $\{U_i : i \in I\}$ of irreducible G-subspaces U_i of V such that $V = \sum_{i \in I} U_i$. We consider the set $\mathcal{I}$ of subsets J of I such that the sum $\sum_{i \in J} U_i$ is direct. The set $\mathcal{I}$ is nonempty; we show it is inductively ordered by inclusion. For, suppose we have a totally ordered set $\{J_a : a \in A\}$ of elements of $\mathcal{I}$. Put $J = \bigcup_{a \in A} J_a$. If the sum $\sum_{j \in J} U_j$ is not direct, there is a finite subset S of J for which $\sum_{j \in S} U_j$ is not direct. Since S must be contained in some J_a, this is impossible. Therefore $J \in \mathcal{I}$. We can now apply Zorn's Lemma to get a maximal element J_0 of $\mathcal{I}$. For this set, we have

$$V = \bigoplus_{a \in J_0} U_i,$$

as required for (2).

In (3), let W be a G-subspace of V. By (2), we can assume that $V = \bigoplus_{i \in I} U_i$, for a family (U_i) of irreducible G-subspaces of V. We consider the set $\mathcal{J}$ of subsets J of I for which $W \cap \sum_{i \in J} U_i = 0$. Again, the set $\mathcal{J}$ is nonempty and inductively ordered by inclusion. If J is a maximal element of $\mathcal{J}$, the sum $X = W + \bigoplus_{j \in J} U_j$ is direct. If $X \neq V$, there is $i \in I$ with $U_i \not\subset X$, so the sum $X + U_i$ is direct, and $J \cup \{i\} \in \mathcal{J}$, contrary to hypothesis. Thus (2) ⇒ (3).

Suppose now that (3) holds. Let V_0 be the sum of all irreducible G-subspaces of V and write $V = V_0 \oplus W$, for some G-subspace W of V.

Assume for a contradiction that $W \neq 0$. By its definition, the space W has no irreducible G-subspace. However, there is a non-zero G-subspace W_1 of W which is finitely generated over G. By Zorn's Lemma, W_1 has a maximal G-subspace W_0, and then W_1/W_0 is irreducible. We have $V = V_0 \oplus W_0 \oplus U$, for some G-subspace U of V, and hence a G-projection $V \to U$. The image of W_1 in U is an irreducible G-subspace of U, hence an irreducible subspace of V which is not contained in V_0. This is nonsense, so $V = V_0$ and (3) $\Rightarrow$ (1). □

One says that (π, V) is *G-semisimple* if it satisfies the conditions of the proposition. Interesting locally profinite groups G usually have many representations which are not semisimple. However, the property can be employed in a slightly different context:

Lemma. *Let G be a locally profinite group, and let K be a compact open subgroup of G. Let (π, V) be a smooth representation of G. The space V is the sum of its irreducible K-subspaces.*

Proof. Let $v \in V$. As in 2.1 Example 2, v is fixed by an open normal subgroup K' of K, and it generates a finite-dimensional K-space W on which K' acts trivially. Thus W is effectively a finite-dimensional representation of the finite group K/K' and so is the sum of its irreducible K-subspaces. Since $v \in V$ was chosen at random, the lemma follows. □

The lemma says that V is *K-semisimple.*

2.3. Again let G be a locally profinite group and K a compact open subgroup of G.

Let $\widehat{K}$ denote the set of equivalence classes of irreducible smooth representations of K. If $\rho \in \widehat{K}$ and (π, V) is a smooth representation of G, we define V^ρ to be the sum of all irreducible K-subspaces of V of class ρ. We call V^ρ the *ρ-isotypic component of* V. In particular, V^K is the isotypic subspace for the class of the trivial representation 1 of K.

Proposition. *Let (π, V) be a smooth representation of G and let K be a compact open subgroup of G.*

(1) *The space V is the direct sum of its K-isotypic components:*

$$V = \bigoplus_{\rho \in \widehat{K}} V^\rho.$$

(2) *Let (σ, W) be a smooth representation of G. For any G-homomorphism $f : V \to W$ and $\rho \in \widehat{K}$, we have*

$$f(V^\rho) \subset W^\rho \quad \text{and} \quad W^\rho \cap f(V) = f(V^\rho).$$

Proof. We use 2.2 to write

$$V = \bigoplus_{i \in I} U_i,$$

for a family of irreducible K-subspaces U_i of V. We let $U(\rho)$ be the sum of those U_i of class ρ. We then have

$$V = \bigoplus_{\rho \in \widehat{K}} U(\rho).$$

If W is an irreducible K-subspace of V of class ρ, then $W \subset U(\rho)$: otherwise, there would be a non-zero K-homomorphism $W \to U_i$, for some U_i of some class $\tau \neq \rho$. We deduce that $V^\rho = U(\rho)$ and (1) follows.

In (2), the image of V^ρ is a sum of irreducible K-subspaces of W, all of class ρ and therefore contained in W^ρ. Moreover, $f(V)$ is the sum of the images $f(V^\tau)$, $\tau \in \widehat{K}$, and $f(V^\tau) \subset W^\tau$. Since the sum of the W^τ is direct, $f(V)$ is the direct sum of the $f(V^\tau)$ and the second assertion follows. □

We frequently use part (2) of the Proposition in the following context:

Corollary 1. *Let $a : U \to V$, $b : V \to W$ be G-homomorphisms between smooth representations U, V, W of G. The sequence*

$$U \xrightarrow{\ a\ } V \xrightarrow{\ b\ } W$$

is exact if and only if

$$U^K \xrightarrow{\ a\ } V^K \xrightarrow{\ b\ } W^K$$

is exact, for every compact open subgroup K of G.

If H is a subgroup of G, we define

$$V(H) = \text{the linear span of } \{v - \pi(h)v : v \in V,\ h \in H\}. \tag{2.3.1}$$

In particular, $V(H)$ is an H-subspace of V.

Corollary 2. *Let G be a locally profinite group, and let (π, V) be a smooth representation of G. Let K be a compact open subgroup of G. Then*

$$V(K) = \bigoplus_{\substack{\rho \in \widehat{K}, \\ \rho \neq 1}} V^\rho, \qquad V = V^K \oplus V(K),$$

and $V(K)$ is the unique K-complement of V^K in V.

Proof. The sum $W = \bigoplus V^\rho$, with ρ not trivial, is a K-complement of V^K in V. There is therefore a K-surjection $V \to V^K$ with kernel W. Clearly, $V(K)$ is contained in the kernel of any K-homomorphism $V \to V^K$; we conclude that W contains $V(K)$. On the other hand, if U is an irreducible K-space of class $\rho \neq 1$, then $U(K) = U$, so $V^\rho \subset V(K)$. □

Exercises.

(1) *Let (π, V) be an* abstract *(i.e., not necessarily smooth) representation of G. Define*

$$V^\infty = \bigcup_K V^K,$$

where K ranges over the compact open subgroups of G. Show that V^∞ is a G-stable subspace of V. Define a homomorphism

$$\pi^\infty : G \longrightarrow \mathrm{Aut}_{\mathbb{C}}(V^\infty)$$

by $\pi^\infty(g) = \pi(g) \mid V^\infty$. Show that (π^∞, V^∞) is a smooth representation of G.

(2) *Let (π, V) be a smooth representation of G and (σ, W) an abstract representation. Let $f : V \to W$ be a G-homomorphism. Show that $f(V) \subset W^\infty$, and hence* $\mathrm{Hom}_G(V, W) = \mathrm{Hom}_G(V, W^\infty)$.

(3) *Let*

$$0 \to U \xrightarrow{\ a\ } V \xrightarrow{\ b\ } W \to 0$$

be an exact sequence of G-homomorphisms of abstract G-spaces. Show that the induced sequence

$$0 \to U^\infty \xrightarrow{\ a\ } V^\infty \xrightarrow{\ b\ } W^\infty$$

is exact. Show by example that the map $b : V^\infty \to W^\infty$ need not be surjective.

2.4. We now consider the notion of an *induced representation.*

Let G be a locally profinite group, and let H be a closed subgroup of G. Thus H is also locally profinite.

Let (σ, W) be a smooth representation of H. We consider the space X of functions $f : G \to W$ which satisfy

(1) $f(hg) = \sigma(h)f(g)$, $h \in H$, $g \in G$;
(2) *there is a compact open subgroup K of G (depending on f) such that $f(gx) = f(g)$, for $g \in G$, $x \in K$.*

We define a homomorphism $\Sigma : G \to \mathrm{Aut}_{\mathbb{C}}(X)$ by

$$\Sigma(g)f : x \longmapsto f(xg), \quad g, x \in G.$$

The pair (Σ, X) provides a smooth representation of G. It is called the representation of G *smoothly induced* by σ, and is usually denoted

$$(\Sigma, X) = \mathrm{Ind}_H^G \, \sigma.$$

The map $\sigma \mapsto \mathrm{Ind}_H^G \, \sigma$ gives a functor $\mathrm{Rep}(H) \to \mathrm{Rep}(G)$.

There is a canonical H-homomorphism

$$\begin{aligned} \alpha_\sigma : \mathrm{Ind}_H^G \, \sigma &\longrightarrow W, \\ f &\longmapsto f(1). \end{aligned}$$

The pair $(\mathrm{Ind}_H^G, \alpha)$ has the following fundamental property:

Frobenius Reciprocity. *Let H be a closed subgroup of a locally profinite group G. For a smooth representation (σ, W) of H and a smooth representation (π, V) of G, the canonical map*

$$\begin{aligned} \mathrm{Hom}_G(\pi, \mathrm{Ind}_H^G \, \sigma) &\longrightarrow \mathrm{Hom}_H(\pi, \sigma), \\ \phi &\longmapsto \alpha_\sigma \circ \phi, \end{aligned}$$

is an isomorphism that is functorial in both variables π, σ.

Proof. Let $f : V \to W$ be an H-homomorphism. We define a G-homomorphism $f_\star : V \to \mathrm{Ind}\,\sigma$ by letting $f_\star(v)$ be the function $g \mapsto f(\pi(g)v)$. The map $f \mapsto f_\star$ is then the inverse of (2.4.2). □

A simple, but very useful, consequence is that $\alpha_\sigma(V) \neq 0$, for any non-zero G-subspace V of $\mathrm{Ind}_H^G \, \sigma$.

We will also need a less formal property:

Proposition. *The functor $\mathrm{Ind}_H^G : \mathrm{Rep}(H) \to \mathrm{Rep}(G)$ is additive and exact.*

Proof. For a smooth representation (σ, W) of H, temporarily let $I(\sigma)$ denote the space of functions $G \to W$ satisfying the first condition $f(hg) = \sigma(h)f(g)$ of the definition above. Thus I is a functor to the category of abstract representations of G; it is clearly additive and exact, while $\mathrm{Ind}_H^G(\sigma) = I(\sigma)^\infty$. Thus Ind_H^G is surely additive, and 2.3 Exercise (3) shows it to be left-exact.

To prove it is right-exact, let (σ, W), (τ, U) be smooth representations of H and let $f : W \to U$ be an H-surjection. Take $\phi \in I(\tau)^\infty$, and choose a compact open subgroup K of G which fixes ϕ. The support of ϕ is a union of cosets HgK, and the value $\phi(g) \in U$ must be fixed by $\tau(H \cap gKg^{-1})$. By 2.3 Corollary 1 (applied to the group H and the trivial representation of its compact open subgroup $H \cap gKg^{-1}$), there exists $w_g \in W$, fixed by $\sigma(H \cap gKg^{-1})$, such that $f(w_g) = \phi(g)$. We define a function $\Phi : G \to W$ to have the same support as ϕ and $\Phi(hgk) = \sigma(h)w_g$, for each $g \in H\backslash \operatorname{supp} \phi/K$. The function Φ is fixed by K, and hence lies in $I(\sigma)^\infty$. Its image in $I(\tau)^\infty$ is ϕ, as required. □

2.5. There is a variation on this theme. With (σ, W) and X as in 2.4, consider the space X_c of functions $f \in X$ which are *compactly supported modulo H*: this means that the image of the support $\operatorname{supp} f$ of f in $H\backslash G$ is compact or, equivalently, $\operatorname{supp} f \subset HC$, for some compact set C in G. The space X_c is stable under the action of G and provides another smooth representation of G. It is denoted $c\text{-}\mathrm{Ind}_H^G \sigma$, and gives a functor

$$c\text{-}\mathrm{Ind}_H^G : \mathrm{Rep}(H) \longrightarrow \mathrm{Rep}(G).$$

One calls it *compact induction,* or *smooth induction with compact supports.*

Exercise 1. *Show that the functor $c\text{-}\mathrm{Ind}_H^G$ is additive and exact.*

In all cases, there is a canonical G-embedding $c\text{-}\mathrm{Ind}_H^G \sigma \to \mathrm{Ind}_H^G \sigma$. Put another way, there is a morphism of functors $c\text{-}\mathrm{Ind}_H^G \to \mathrm{Ind}_H^G$. This is an isomorphism if and only if $H\backslash G$ is compact. On the other hand, for specific G, H, σ, the map $c\text{-}\mathrm{Ind}_H^G \sigma \to \mathrm{Ind}_H^G \sigma$ can be an isomorphism even when $H\backslash G$ is not compact. Significant examples of this phenomenon arise in 11.4 below.

This construction is mainly (but not exclusively) of interest when the subgroup H is *open* in G. In this case, there is a canonical H-homomorphism

$$\begin{aligned} \alpha_\sigma^c : W &\longrightarrow c\text{-}\mathrm{Ind}\,\sigma, \\ w &\longmapsto f_w, \end{aligned} \tag{2.5.1}$$

where $f_w \in X_c$ is supported in H and $f_w(h) = \sigma(h)w$, $h \in H$.

Exercise 2. *Suppose H is open in G. Let $\phi : G \to W$ be a function, compactly supported modulo H, such that $\phi(hg) = \sigma(h)\phi(g)$, $h \in H$, $g \in G$. Show that $\phi \in X_c$.*

Lemma. *Let H be an open subgroup of G, and let (σ, W) be a smooth representation of H.*

(1) *The map $\alpha_\sigma^c : w \mapsto f_w$ is an H-isomorphism of W with the space of functions $f \in c\text{-}\mathrm{Ind}_H^G \sigma$ such that* $\operatorname{supp} f \subset H$.
(2) *Let $\mathcal{W}$ be a $\mathbb{C}$-basis of W and $\mathcal{G}$ a set of representatives for G/H. The set $\{gf_w : w \in \mathcal{W}, g \in \mathcal{G}\}$ is a $\mathbb{C}$-basis of $c\text{-}\mathrm{Ind}\,\sigma$.*

Proof. In (1), surely α_σ^c is an H-homomorphism to the space of functions supported in H; the inverse map is $f \mapsto f(1)$.

The support of a function $f \in c\text{-Ind}_H^G \sigma$ is a finite union of cosets Hg^{-1}, for various $g \in \mathcal{G}$, and the restriction of f to any one of these also lies in $c\text{-Ind}\,\sigma$. If $\operatorname{supp} f = Hg^{-1}$, then $g^{-1}f$ has support contained in H, and so is a finite linear combination of functions f_w, $w \in \mathcal{W}$. Clearly, the set of functions gf_w, $w \in \mathcal{W}$, $g \in \mathcal{G}$, is linearly independent, and the proof is complete. □

For open subgroups, compact induction has its own form of Frobenius Reciprocity property:

Proposition. *Let H be an open subgroup of G, let (σ, W) be a smooth representation of H and (π, V) a smooth representation of G. The canonical map*

$$\begin{aligned} \operatorname{Hom}_G(c\text{-Ind}\,\sigma, \pi) &\longrightarrow \operatorname{Hom}_H(\sigma, \pi), \\ f &\longmapsto f \circ \alpha_\sigma^c, \end{aligned}$$

is an isomorphism which is functorial in both variables.

Proof. Let ϕ be an H-homomorphism $W \to V$. There is a unique G-homomorphism $\phi_* : c\text{-Ind}\,\sigma \to V$ such that $\phi_*(f_w) = \phi(w)$, $w \in W$. The map $\phi \mapsto \phi_*$ is then inverse to (2.5.2). □

Remark. Suppose for the moment that G is *finite*. The coincident definitions of induction above are then equivalent to the standard one: this is easily proved directly. Alternatively, the version (2.5.2) of Frobenius Reciprocity is the same as the usual one for finite groups and one can use uniqueness of adjoint functors.

2.6. It is convenient to introduce a technical restriction on the group G. *From now on, we assume that:*

Hypothesis. *For any compact open subgroup K, the set G/K is countable.*

We remark that, if G/K is countable for *one* compact open subgroup K of G, then G/K' is countable for *any* compact open subgroup K' of G. For, $K \cap K'$ is compact, open, and of finite index in K. Thus the surjection $G/(K \cap K') \to G/K$ has finite fibres and $G/(K \cap K')$, hence also G/K', is countable.

Certain of the things we do in this section do not require the property, but it holds for every concrete group in which we shall be interested.

The main effect of the hypothesis is:

Lemma. *Let (π, V) be an irreducible smooth representation of G. The dimension $\dim_{\mathbb{C}} V$ is countable.*

Proof. Let $v \in V$, $v \neq 0$, and choose a compact open subgroup K of G such that $v \in V^K$. Since V is irreducible, the countable set $\{\pi(g)v : g \in G/K\}$ spans V. □

This enables us to generalize a familiar result, as follows.

Schur's Lemma. *If (π, V) is an irreducible smooth representation of G, then* $\mathrm{End}_G(V) = \mathbb{C}$.

Proof. Let $\phi \in \mathrm{End}_G(V)$, $\phi \neq 0$. The image and the kernel of ϕ are both G-subspaces of V, so ϕ is bijective and invertible. Therefore $\mathrm{End}_G(V)$ is a complex division algebra.

If we fix $v \in V$, $v \neq 0$, the G-translates of v span V so an element $\phi \in \mathrm{End}_G(V)$ is determined uniquely by the value $\phi(v)$. We deduce that $\mathrm{End}_G(V)$ has countable dimension. However, any $\phi \in \mathrm{End}_G(V)$, $\phi \notin \mathbb{C}$, is transcendental over $\mathbb{C}$, and generates a field $\mathbb{C}(\phi) \subset \mathrm{End}_G(V)$. The subset $\{(\phi-a)^{-1} : a \in \mathbb{C}\}$ of $\mathbb{C}(\phi)$ is linearly independent over $\mathbb{C}$, so the $\mathbb{C}$-dimension of $\mathbb{C}(\phi)$ is uncountable, and this is impossible. The only conclusion is that $\mathrm{End}_G(V) = \mathbb{C}$, as required. □

Corollary 1. *Let (π, V) be an irreducible smooth representation of G. The centre Z of G then acts on V via a character $\omega_\pi : Z \to \mathbb{C}^\times$, that is, $\pi(z)v = \omega_\pi(z)v$, for $v \in V$ and $z \in Z$.*

Proof. By Schur's Lemma, there is surely a homomorphism $\omega_\pi : Z \to \mathbb{C}^\times$ such that $\pi(z)v = \omega_\pi(z)v$, $z \in Z$, $v \in V$. If K is a compact open subgroup of G such that $V^K \neq 0$, then ω_π is trivial on the compact open subgroup $K \cap Z$ of Z. Thus ω_π is a character of Z. □

One calls ω_π the *central character* of π.

Corollary 2. *If G is abelian, any irreducible smooth representation of G is one-dimensional.*

Remark. If G is compact, the converse of Schur's Lemma holds: a smooth representation (π, V) of G is a direct sum of irreducible representations, so $\mathrm{End}_G(V)$ is one-dimensional if and only if π is irreducible. This is *false* for smooth representations of locally profinite groups in general: see 9.10 below for a significant example.

2.7. We will sometimes need a more general version of the preceding machinery. Before dealing with this, however, it is convenient to interpolate a general lemma.

Lemma. *Let G be a locally profinite group, and let H be an open subgroup of G of finite index.*

(1) *If* (π, V) *is a smooth representation of* G*, then* V *is* G*-semisimple if and only if it is* H*-semisimple.*
(2) *Let* (σ, W) *be a semisimple smooth representation of* H*. The induced representation* $\mathrm{Ind}_H^G\,\sigma$ *is* G*-semisimple.*

Proof. Suppose that V is H-semisimple, and let U be a G-subspace of V. By hypothesis, there is an H-subspace W of V such that $V = U \oplus W$. Let $f : V \to U$ be the H-projection with kernel W. Consider the map

$$f^G : v \longmapsto (G{:}H)^{-1} \sum_{g \in G/H} \pi(g)\, f(\pi(g)^{-1}v), \quad v \in V.$$

The definition is independent of the choice of coset representatives and it follows that f^G is a G-projection $V \to U$. We then have $V = U \oplus \mathrm{Ker}\, f^G$ and $\mathrm{Ker}\, f^G$ is a G-subspace of V. Thus V is G-semisimple (*cf.* 2.2 Proposition).

Conversely, suppose that V is G-semisimple. Thus G is a direct sum of irreducible G-subspaces (2.2), and it is enough to treat the case where V is irreducible over G. As representation of H, the space V is finitely generated and so admits an irreducible H-quotient U. Suppose for the moment that H is a *normal* subgroup of G. By Frobenius Reciprocity (2.4.2), the H-map $V \to U$ gives a non-trivial, hence injective, G-map $V \to \mathrm{Ind}_H^G\, U$. As representation of H, the induced representation $\mathrm{Ind}_H^G\, U = c\text{-}\mathrm{Ind}_H^G\, U$ is a direct sum of G-conjugates of U (*cf.* 2.5 Lemma). These are all irreducible over H, so $\mathrm{Ind}\, U$ is H-semisimple. Proposition 2.2 then implies that $V \subset \mathrm{Ind}\, U$ is H-semisimple.

In general, we set $H_0 = \bigcap_{g \in G/H} gHg^{-1}$. This is an open normal subgroup of G of finite index. We have just shown that the G-space V is H_0-semisimple; the first part of the proof shows it is H-semisimple.

This completes the proof of (1), and (2) follows readily from the same arguments. □

We first apply this in the following context. Let Z be the centre of G, and fix a character χ of Z. We consider the class of smooth representations (π, V) of G which *admit* χ *as central character,* that is,

$$\pi(z)v = \chi(z)v, \quad v \in V,\ z \in Z.$$

Proposition. *Let* (π, V) *be a smooth representation of* G*, admitting* χ *as a central character. Let* K *be an open subgroup of* G *such that* KZ/Z *is compact.*

(1) *Let* $v \in V$*. The* KZ*-space spanned by* v *is of finite dimension, and is a sum of irreducible* KZ*-spaces.*
(2) *As representation of* KZ*, the space* V *is semisimple.*

Proof. The vector v is fixed by a compact open subgroup K_0 of K. The set KZ/K_0Z is finite, so the space W spanned by $\pi(KZ)v$ has finite dimension. Surely W is K_0Z-semisimple; the lemma implies it is KZ-semisimple. Since v was chosen arbitrarily, assertion (2) follows. □

In practice, the open subgroup K will contain Z, with K/Z compact. The discussion is equally valid if Z is a closed subgroup of the centre $Z(G)$ of G.

2.8. The notion of *duality,* for smooth representations of a locally profinite group, is both more subtle and more significant than it is for representations of finite groups. We examine it in some detail.

Let (π, V) be a smooth representation of the locally profinite group G. Write $V^* = \mathrm{Hom}_{\mathbb{C}}(V, \mathbb{C})$, and denote by

$$\begin{aligned} V^* \times V &\longrightarrow \mathbb{C}, \\ (v^*, v) &\longmapsto \langle v^*, v\rangle, \end{aligned}$$

the canonical evaluation pairing. The space V^* carries a representation π^* of G defined by

$$\langle \pi^*(g)v^*, v\rangle = \langle v^*, \pi(g^{-1})v\rangle, \quad g \in G,\ v^* \in V^*,\ v \in V.$$

This is not, in general, smooth. We accordingly define

$$\check{V} = (V^*)^\infty = \bigcup_K (V^*)^K,$$

where K ranges over the compact open subgroups of G. Thus (*cf.* 2.3 Exercise (1)) $\check{V}$ is a G-stable subspace of V^*, and provides a smooth representation

$$\check{\pi} = (\pi^*)^\infty : G \longrightarrow \mathrm{Aut}_{\mathbb{C}}(\check{V}).$$

The representation $(\check{\pi}, \check{V})$ is called the *contragredient,* or *smooth dual,* of (π, V). We continue to denote the evaluation pairing $\check{V} \times V \to \mathbb{C}$ by $(\check{v}, v) \mapsto \langle \check{v}, v\rangle$. Therefore

$$\langle \check{\pi}(g)\check{v}, v\rangle = \langle \check{v}, \pi(g^{-1})v\rangle, \quad g \in G,\ \check{v} \in \check{V},\ v \in V. \tag{2.8.1}$$

Let K be a compact open subgroup of G. We recall that V^K has a unique K-complement $V(K)$ in V (2.3). If $\check{v} \in \check{V}$ is fixed under K, we have $\langle \check{v}, V(K)\rangle = 0$, by the definition of $V(K)$. Thus $\check{v} \in \check{V}^K$ is determined by its effect on V^K.

Proposition. *Restriction to V^K induces an isomorphism $\check{V}^K \cong (V^K)^*$.*

Proof. One can extend a linear functional on V^K to an element of $\check{V}^K$ by deeming that it be trivial on $V(K)$. □