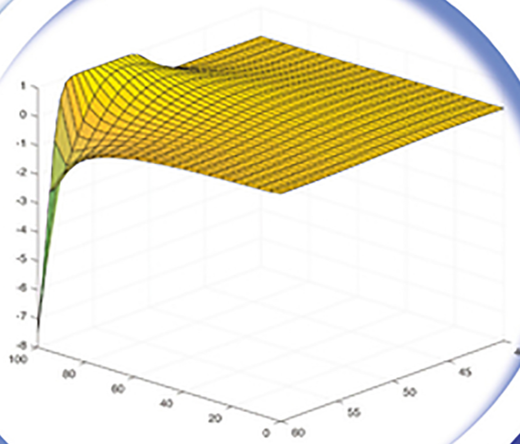
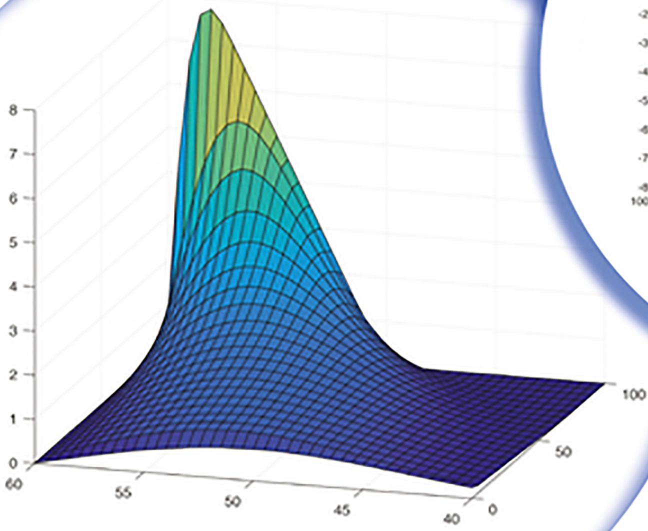
An Introduction to

FINANCIAL MARKETS

A Quantitative Approach





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Preface

This book arises from slides and lecture notes that I have used over the years in my courses *Financial Markets and Instruments* and *Financial Engineering*, which were offered at Politecnico di Torino to graduate students in Mathematical Engineering. Given the audience, the treatment is naturally geared toward a mathematically inclined reader. Nevertheless, the required prerequisites are relatively modest, and any student in engineering, mathematics, and statistics should be well-equipped to tackle the contents of this introductory book.[1] The book should also be of interest to students in economics, as well as junior practitioners with a suitable quantitative background.

We begin with quite elementary concepts, and material is introduced progressively, always paying due attention to the practical side of things. Mathematical modeling is an art of selective simplification, which must be supported by intuition building, as well as by a healthy dose of skepticism. This is the aim of remarks, counterexamples, and financial horror stories that the book is interspersed with. Occasionally, we also touch upon current research topics.

Book structure

The book is organized into five parts.

1. Part One, **Overview**, consists of two chapters. Chapter 1 aims at getting unfamiliar readers acquainted with the role and structure of financial markets, the main classes of traded assets (equity, fixed income, and derivatives), and the main types of market participants, both in terms of institutions (e.g., investments banks and pension funds) and roles (e.g., speculators, hedgers, and arbitrageurs). We try to give a practical flavor that is essential to students of quantitative disciplines, setting the stage for the application of quantitative models. Chapter 2 overviews the basic problems in finance, like asset allocation, pricing, and risk management, which may be tackled by quantitative models. We also introduce the fundamental concepts related to arbitrage theory, including market completeness and risk-neutral measures, in a simple static and discrete setting.
2. Part Two, **Fixed-income assets**, consists of four chapters and introduces the simplest assets depending on interest rates, starting with plain bonds. The fundamental concepts of interest rate modeling, including the term

[1] In case of need, the mathematical prerequisites are covered in my other book: *Quantitative Methods: An Introduction for Business Management*. Wiley, 2011.

structure and forward rates, as well as bond pricing, are covered in Chapter 3. The simplest interest rate derivatives (forward rate agreements and vanilla swaps) are covered in Chapter 4, whereas Chapter 5 aims at providing the reader with a flavor of real-life markets, where details like day count and quoting conventions are relevant. Chapter 6 concludes this part by showing how quantitative models may be used to manage interest rate risk. In this part, we do not consider interest rate options, which require a stronger mathematical background and are discussed later.

3. Part Three, **Equity portfolios**, consists of four chapters, where we discuss equity markets and portfolios of stock shares. Actually, this is not the largest financial market, but it is arguably the kind of market that the layman is more familiar with. Chapter 7 is a bit more theoretical and lays down the foundations of static decision-making under uncertainty. By static, we mean that we make one decision and then we wait for its consequences, finger crossed. Multistage decision models are discussed later. In this chapter, we also introduce the basics of risk aversion and risk measurement. Chapter 8 is quite classical and covers traditional mean–variance portfolio optimization. The impact of statistical estimation issues on portfolio management motivates the introduction of factor models, which are the subject of Chapter 9. Finally, in Chapter 10, we discuss equilibrium models in their simplest forms, the capital asset pricing model (CAPM), which is related to a single-index factor model, and arbitrage pricing theory (APT), which is related to a multifactor model. We do not discuss further developments in equilibrium models, but we hint at some criticism based on behavioral finance.
4. Part Four, **Derivatives**, includes four chapters. We discuss dynamic uncertainty models in Chapter 11, which is more challenging than previous chapters, as we have to introduce the necessary foundations of option pricing models, namely, stochastic differential equations and stochastic integrals. Chapter 12 describes simple forward and futures contracts, extending concepts that were introduced in Chapter 4, when dealing with forward and futures interest rates. Chapter 13 covers option pricing in the case of complete markets, including the celebrated and controversial Black–Scholes–Merton formula, whereas Chapter 14 extends the basic concepts to the more realistic setting of incomplete markets.
5. Part Five, **Advanced optimization models**, is probably the less standard part of this book, when compared to typical textbooks on financial markets. We deal with optimization model building, in Chapter 15, and optimization model solving, in Chapter 16. Actually, it is difficult to draw a sharp line between model building and model solving, but it is a fact of life that advanced software is available for solving quite sophisticated models, and the average user does not need a very deep knowledge of the involved algorithms, whereas she must be able to build a model. This is the motivation for separating the two chapters.

Needless to say, the choice of which topics should be included or omitted is debatable and based on authors' personal bias, not to mention the need to keep a book size within a sensible limit. With respect to introductory textbooks on financial markets, there is a deeper treatment of derivative models. On the other hand, more challenging financial engineering textbooks do not cover, e.g., equilibrium models and portfolio optimization. We aim at an intermediate treatment, whose main limitations include the following:

- We only hint at criticism put forward by behavioral finance and do not cover market microstructure and algorithmic trading strategies.
- From a mathematical viewpoint, we pursue an intuitive treatment of financial engineering models, as well as a simplified coverage of the related tools of stochastic calculus. We do not rely on rigorous arguments involving self-financing strategies, martingale representation theorems, or change of probability measures.
- From a financial viewpoint, by far, the most significant omission concerns credit risk and credit derivatives. Counterparty and liquidity risk play a prominent role in post-Lehman Brothers financial markets and, as a consequence of the credit crunch started in 2007, new concepts like CVA, DVA, and FVA have been introduced. This is still a field in flux, and the matter is arguably not quite assessed yet.
- Another major omission is econometric time series models.

Adequate references on these topics are provided for the benefit of the interested readers.

My choices are also influenced by the kind of students this book is mainly aimed at. The coverage of optimization models and methods is deeper than usual, and I try to open readers' critical eye by carefully crafted examples and counterexamples. I try to strike a satisfactory balance between the need to illustrate mathematics in action and the need to understand the real-life context, without which quantitative methods boil down to a solution in search of a problem (or a hammer looking for nails, if you prefer). I also do not disdain just a bit of repetition and redundancy, when it may be convenient to readers who wish to jump from chapter to chapter. More advanced sections, which may be safely skipped by readers, are referred to as *supplements* and their number is marked by an initial "S."

In my Financial Engineering course, I also give some more information on numerical methods. The interested reader might refer to my other books:

- P. Brandimarte, *Numerical Methods in Finance and Economics: A MATLAB-Based Introduction* (2nd ed.), Wiley, 2006
- P. Brandimarte, *Handbook in Monte Carlo Simulation: Applications in Financial Engineering, Risk Management, and Economics*, Wiley, 2014

Acknowledgements

In the past years, I have adopted the following textbooks (or earlier editions) in my courses. I have learned a lot from them, and they have definitely influenced the writing of this book:

- Z. Bodie, A. Kane, and A. Marcus, *Investments* (9th ed.), McGraw-Hill, 2010
- J.C. Hull, *Options, Futures, and Other Derivatives* (8th ed.), Prentice Hall, 2011
- P. Veronesi, *Fixed Income Securities: Valuation, Risk, and Risk Management*, Wiley, 2010

Other specific acknowledgements are given in the text. I apologize in advance for any unintentional omission.

Additional material

Some end-of-chapter problems are included and fully worked solutions will be posted on a web page. My current URL is

- `http://staff.polito.it/paolo.brandimarte/`

A hopefully short list of errata will be posted there as well. One of the many corollaries of Murphy's law states that my URL is going to change shortly after publication of the book. An up-to-date link will be maintained on the Wiley web page:

- `http://www.wiley.com/`

For comments, suggestions, and criticisms, all of which are quite welcome, my e-mail address is

- `paolo.brandimarte@polito.it`

PAOLO BRANDIMARTE

Turin, September 2017

About the Companion Website

This book is accompanied by a companion website:

www.wiley.com/go/brandimarte/financialmarkets

The website includes:

- Solutions manual for end-of-chapter problems

Part One

Overview

Chapter One

Financial Markets: Functions, Institutions, and Traded Assets

Providing a simple, yet exhaustive definition of finance is no quite easy task, but a possible attempt, at least from a conceptual viewpoint, is the following:[1]

> *Finance is the study of how people and organizations allocate scarce resources over time, subject to uncertainty.*

This definition might sound somewhat generic, but it does involve the two essential ingredients that we shall deal with in practically every single page of this book: *Time* and *uncertainty*. Appreciating their role is essential in understanding why finance was born in the past and is so pervasive now. The time value of money is reflected in the interest rates that define how much money we have to pay over the time span of our mortgage, or the increase in wealth that we obtain by locking up our capital in a certificate of deposit issued by a bank. It is common wisdom that the value of $1 now is larger than the value of $1 in one year. This is not only a consequence of the potential loss of value due to inflation.[2] A dollar now, rather than in the future, paves the way to earlier investment opportunities, and it may also serve as a precautionary cushion against unforeseen needs. Uncertainty is related, e.g., to the impossibility of forecasting the return that we obtain from investing in stock shares, but also to the risk of adverse movements in currency exchange rates for an import/export firm, or longevity risk for a worker approaching retirement. As we show in Chapter 2, we may model issues related to time and uncertainty within a mathematical framework, applying principles from financial economics and tools from probability, statistics, and optimization theory. Before doing that, we need a more concrete view

[1] This definition is taken from [2].

[2] This holds under common economic conditions; the exception to the rule is deflation, which is (at the time of writing) a possibility in Euroland. In this book, we will assume that the standard economic conditions prevail.

in order to understand how financial markets work, which kinds of assets are exchanged, and which actors play a role in them and what their incentives are. We pursue this "institutional" approach to get acquainted with finance in this chapter. Some of the more mathematically inclined students tend to consider this side of the coin modestly exciting, but a firm understanding of it is necessary to put models in the right perspective and to appreciate their pitfalls and limitations.

In Section 1.1, we discuss the role of time and uncertainty in a rather abstract way that, nevertheless, lays down some essential concepts. A more concrete view is taken in Section 1.2, where we describe the fundamental classes of assets that are traded on financial markets, namely, stock shares, bonds, currencies, and the basic classes of derivatives, like forward/futures contracts and options. In order to provide a proper framework, we also hint at the essential shape of a balance sheet, in terms of assets, liabilities, and equity, and we emphasize the difference between standardized assets traded on regulated exchanges and less liquid assets, possibly engineered to meet specific client requirements, which are traded over-the-counter. In Section 1.3, we describe the classes of players involved in financial markets, such as investment/commercial banks, common/hedge/pension funds, insurance companies, brokers, and dealers. We insist on the separation between the institutional form and the role of those players: A single player may be of one given kind, in institutional terms, but it may play different roles. For instance, an investment bank can, among many other things, play the role of a prime broker for a hedge fund. Furthermore, depending on circumstances, players may act as hedgers, speculators, or arbitrageurs. The exact organization of financial markets is far from trivial, especially in the light of extensive use of information technology, and a full description is beyond the scope of this book. Nevertheless, some essential concepts are needed, such as the difference between primary and secondary markets, which is explained in Section 1.4. There, we also introduce some trading strategies, like buying on margin and short-selling, which are essential to interpret what happens on financial markets in practice, as well as to understand some mathematical arguments that we will use over and over in this book. Finally, in Section 1.5 we consider market indexes and describe some basic features explaining, for instance, the difference between an index like the Dow Jones Industrial Average and the Standard & Poor 500.

1.1 What is the purpose of finance?

If you are reading this book, chances are that it is because you would like to land a rewarding job in finance. Even if this is not the case, one of the reasons why we aim at finding a good job is because we need to earn some income in order to purchase goods and services, for ourselves and possibly other people we care about. Every month (hopefully) we receive some income, and we must plan its use. The old grasshopper and ant fable teaches that we should actually

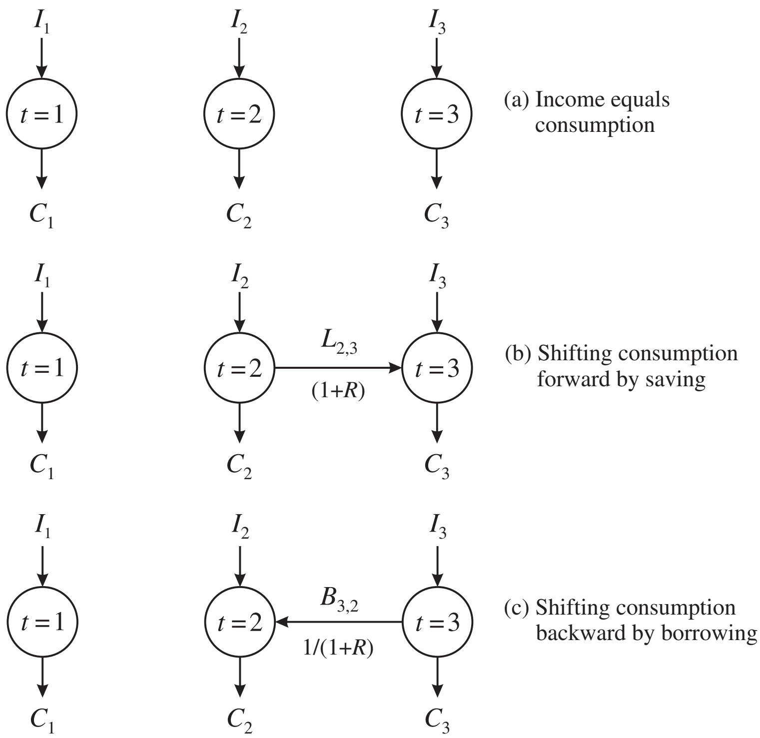

FIGURE 1.1 **Shifting consumption forward and backward in time.**

plan ahead with care. Part of that income should be saved to allow consumption at some later time. Sometimes, we might need to use more income than we are earning at present, e.g., in order to finance the purchase of our home sweet home.

Now, imagine a world in which we cannot "store" money, and we have to consume whatever our income is *immediately*, no more, no less, just like we would do with perishable food, if no one had invented refrigerators and other conservation techniques. This unpleasing situation is depicted in Fig. 1.1(a). There, time is discretized in $T = 3$ time periods, indexed by $t = 1, \dots, T$.[3] The income during time period t is denoted by I_t, and it is equal to the consumption C_t during the same period:

$$I_t = C_t, \qquad t = 1, \dots, T.$$

[3]Sometimes, time discretization requires careful thinking about events. Do we earn income at the beginning or at the end of a time period? In other words, is income earned during time period t immediately available for consumption during the same time period? We may argue that income during time period t is available for consumption only during time period $t+1$. We shall discuss more precise notation and concepts in Section 2.1.2. Here, for the sake of simplicity, we assume that every event during a time *period* is concentrated at some time *instant*. We sometimes use the rather awkward term *epoch* to refer to a specific point in time. We also often use the term time *bucket* to refer to a time period delimited by two time instants.

This state of the matter is not quite satisfactory, if we have excess income in some period and would like to delay consumption to a later time period. In Fig. 1.1(b), part of income I_2, denoted by $L_{2,3}$, is shifted forward from time period 2 to time period 3. This results in an increase of C_3 and a decrease of C_2. The amount of income saved can be regarded as money invested or lent to someone else. By a similar token, we might wish to anticipate consumption to an earlier time period. In Fig. 1.1(c), consumption C_2 is increased by shifting income backward in time from time period $t = 3$, which means borrowing an amount of money $B_{3,2}$, to be used in time period $t = 2$ and repaid in time period $t = 3$. Savers and borrowers may be individuals or institutions, and we may play both of these roles at different stages of our working life. Clearly, all of this may happen if there is a way to match savers and borrowers, so that all of them may improve their consumption timing. This is one of the many roles of financial markets; more specifically, we use the term **money markets** when the time span of the loan is short. In other cases, the investment may stretch over a considerable time span, especially if savers/borrowers are not just households, but corporations, innovative startups, or public administrations that have to finance the development of a new product, the building of a new hospital, or an essential infrastructure. In this case, we talk about **capital markets**.

Needless to say, if we accept to delay consumption, it is because we expect to be compensated in some way. Informally, we exchange an egg for a chicken; formally, we earn some interest rate R along the time period involved in the shift.[4] We may interpret the shift as a flow of money over a network in time but, unlike other network flows involved in transportation over space, we do not have exact conservation of flows. With reference to Fig. 1.1(b), we have the following flow balance equations at nodes 2 and 3:

$$C_2 = I_2 - L_{2,3},$$
$$C_3 = I_3 + L_{2,3}(1 + R),$$

stating that we give up an amount $L_{2,3}$ of consumption at time 2 in exchange for an increase $(1 + R)L_{2,3}$ in later consumption. The factor $1 + R$ is a gain associated with the flow of money along the arc connecting node $t = 2$ to node $t = 3$. This is what the **time value** of money is all about. The exact value of the interest rate R, as we shall see in Chapter 3, may be related to the possibility of default (i.e., the borrower may not repay the full amount of his debt) and to inflation risk, among other things.

Clearly, there must be another side of the coin: The increase in later consumption must be paid by a counterparty in an exchange. We delay consumption while someone else anticipates it. With reference to Fig. 1.1(c), we have

[4] In financial practice, whenever an interest rate is quoted, it is always an *annual* rate. For now, let us associate the rate with an arbitrary time period.

the following flow balance equations at nodes 2 and 3:

$$C_2 = I_2 + \frac{B_{3,2}}{1+R},$$
$$C_3 = I_3 - B_{3,2}.$$

Note that we are expressing the borrowed amount $B_{3,2}$ in terms of the money at time $t = 3$, when the debt is repaid; in other words $B_{3,2}$ is a flow out of node 3. This is not essential at all: If we use money at time $t = 2$, i.e., we consider the flow $B^*_{3,2}$ *into* node 2, the flow balance would simply read

$$C_2 = I_2 + B^*_{3,2},$$
$$C_3 = I_3 - B^*_{3,2}(1+R).$$

The two sides of the coin must be somehow matched by a market mechanism. In practice, funds are channeled by financial intermediaries, which must be compensated for their job. In fact, there is a difference between lending and borrowing rates, called **bid–ask** (or bid–offer) **spread**, which applies to other kinds of financial assets as well. Lending and borrowing money through a bank is what we are familiar with as individuals, whereas a large corporation and a sovereign government have the alternative of raising funds by issuing securities like bonds, typically promising the payment of periodic interest, as well as the refund of the capital at some prespecified point in time, the maturity of the bond. Corporations may also raise funds by issuing stock shares. Buying a stock share does not mean that we lend money to a firm; hence, we are not entitled to the payment of any interest. Rather, we own a share of the firm and may receive a corresponding share of earnings that may be distributed in the form of dividends to stockholders. However, the amount that we will receive is random and no promise is made about dividends, as they depend on how well the business is doing, as well as the decision of reinvesting part of the earning in new business ventures, rather than distributing the whole of it.

After being first issued, securities like bonds and stock shares may be exchanged among market participants, at prices that may depend on several underlying risk factors. Since the values of these factors are not known with certainty, the future prices of bonds and stock shares are random. In fact, time is intertwined with another fundamental dimension in finance, namely, **uncertainty**. When we lend or borrow money at a given interest rate, the future cash flows are known with certainty, if we do not consider the possibility of a default on debt. However, when we buy a stock share at time $t = 0$ and plan to sell it at time $t = T$, randomness comes into play. Let us denote the initial price by $S(0)$.[5] The future price $S(T)$ is a **random variable**, which we may denote as $S(T, \omega)$ to emphasize its dependence on the random outcome (scenario) ω. We recall that, in probability theory, a random variable is a function mapping underlying random outcomes, corresponding to future scenarios or states of nature, to numeric values. Let ω_i, $i = 1, \ldots, m$, denote the i-th outcome, which

[5] Depending on notational convenience, we shall write $S(t)$ or S_t, as no ambiguity should arise.

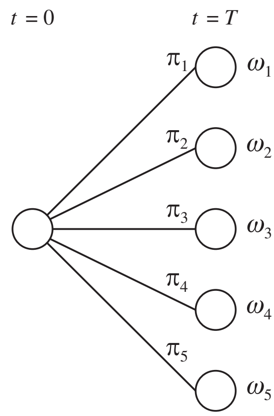

FIGURE 1.2 **Representing uncertain states of the world by a scenario fan.**

occurs with probability π_i. For the sake of simplicity, we are considering a discrete and finite set of possible outcomes, whereas later we will deal extensively with continuous random variables. A simple way to depict this kind of discrete uncertainty is by a scenario fan like the one depicted in Fig. 1.2. Therefore, $S(T, \omega)$ is a random variable, and we associate a future price $S(T, \omega_i)$ with each future state of the world. The corresponding holding period return is defined as follows.

DEFINITION 1.1 (Holding period return) *Let us consider a holding period* $[0, T]$*, where the initial asset price is* $S(0)$ *and the terminal random asset price is* $S(T, \omega)$*. We define the* **holding period *return*** *as*

$$R(\omega) \doteq \frac{S(T, \omega) - S(0)}{S(0)} \tag{1.1}$$

and the **holding period *gain*** *as*

$$G(\omega) \doteq \frac{S(T, \omega)}{S(0)} = 1 + R(\omega). \tag{1.2}$$

The gain and the holding period return (return for short) are clearly related. A return of 10% means that the stock price was multiplied by a gain factor of 1.10.

Remark. The term *gain* is not so common in finance textbooks. Usually, terms like *total return* or *gross return* are used, rather than gain. On the contrary, terms like *rate of return* and *net return* are used to refer to (holding period) return. The problem is that these terms may ring different bells, especially to practitioners. We may use the qualifier "total" when we want to emphasize a return including dividend income, besides the capital gain related to price changes. Terms like "gross" and "net" may be related with taxation issues, which we shall always disregard. This is why we prefer using "gain," even though this usage is less common. We shall not confuse gain, which is a multiplicative factor,