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Foreword

Subdivision surface is a popular modeling technique in the field of computer-aided design (CAD) and computer graphics (CG) for its strong modeling capabilities for meshes of any topology. This book makes a comprehensive introduction to subdivision modeling technologies, the focus of which lies in not only fundamental theories but also practical applications. In theory aspect, this book seeks to make readers understand the contacts between spline surfaces and subdivision surfaces and makes the readers master the analysis techniques of subdivision surfaces. In application aspect, it introduces some typical modeling techniques, such as interpolation, fitting, fairing, intersection, trimming and interactive edit. By studying this book, readers can grasp the main technologies of subdivision surface modeling and use them in software development. This knowledge also benefits understanding of CAD/CG software operations.

Due to flexible topology adaptivity and strong modeling capability, subdivision surface modeling technology has developed quickly in the field of CAD, CG, and geometric modeling since its appearance during the 1970s. Many famous 3D modeling software, such as 3DMax, Maya, and Meshlab, has involved subdivision surface as a modeling tool. Subdivision modeling technology has been successfully applied in making characters of games and special effects of movies. As the saying goes, "Give a man a fish; you have fed him for today. Teach a man to fish; and you have fed him for a lifetime." On the one hand, the book has done a detailed exposition to the basic theory of subdivision surfaces and strives to make readers to achieve the mastery. On the other hand, although the contents of the book are limited, we make a remarks about the main topic at the end of each chapter and list the closely related references for readers to self-improve. We believe that by learning through this book, readers will have a capability of researching and developing with subdivision surfaces independently.

The book was planned by Prof. Wenhe Liao, and most materials came from doctorial dissertations supervised by him. Associate Professor Hao Liu complied this book and wrote Chaps. 1–6 and Sects. 10.1 and 10.2. Dr. Tao Li arranged the rest of the book. He wrote Chap. 8, and Sects. 7.3, 10.3 and revised Sects. 7.1, 7.2 and Chaps. 9, 11. Chapter 9 was taken from Gang He's doctorial dissertation, and

Dr. He revised the English manuscript of this chapter. Sections 7.1 and 7.2 and Chap. 11 came from Xiangyu Zhang's doctorial dissertation, and Dr. Zhang revised the corresponding English manuscript. Graduate Wei Fan made a lot of work for the final proof. The authors thank Dr. He, Dr. Zhang and Graduate Wei Fan for their contributions to this book.

This book is suitable for graduate students, teachers, and technical personals majoring in CAGD, CAD/CG, and other related fields as a reference book on surface modeling. Due to the limitation of our knowledge, there are inevitably some drawbacks in this book. If any flaw found, we are grateful for your contact with us (liuhao-01@nuaa.edu.cn).

Table of Symbols

Notation

Our general approach to notation is to accord to traditional convention meanwhile precisely to express our intentions. Consequently, we try to use traditional notation as far as possible. At the same time, some special notation is introduced; for example, $\boldsymbol{M}[i,j]$ denotes the entry of a matrix $\boldsymbol{M}$ in the ith row and in the jth row. The highlights of this notation are the following:

- Function application is denoted using parentheses (), for example $\boldsymbol{p}(u)$, $\boldsymbol{p}(u,v)$; a combinational number is also be denoted using parentheses (), for example $\begin{pmatrix} 5 \\ 3 \end{pmatrix} = \frac{5!}{3!(5-3)!}$. Diploid or triple is also denoted using parentheses (), for example $K = (V, E, F)$.
- Vectors and matrices are created by enclosing their members in square brackets, for example $\boldsymbol{U} = [u_0.u_1, u_2, \cdots]$, $\boldsymbol{M} = \begin{bmatrix} v & e & f \\ 0 & 0 & e \\ 0 & e & 0 \end{bmatrix}$. These members are scalars. Conversely, the ith entry of the vector $\boldsymbol{U}$ is denoted by $\boldsymbol{U}[i]$. The entry in the ith row and in the jth row is denoted by $\boldsymbol{M}[i,j]$. If members of a vector are also vectors, we especially denote the vector as $\overrightarrow{\bullet}$. For example $\overrightarrow{\boldsymbol{E}} = \begin{bmatrix} \boldsymbol{e}_0 \\ \boldsymbol{e}_1 \\ \vdots \\ \boldsymbol{e}_{n-1} \end{bmatrix}$. When a vector denotes a coordinate of a point, we also directly use name of components. For example for $\boldsymbol{R} = [x,y,z]$, $\boldsymbol{R}[x]$ denote the x component. We also use a vector to denote a form of a Lave tiling, for example [4,6,12] Lave tiling.
- Sets are created by enclosing their members in curly brackets $\{\}$. We arrange that indices of members of a vector, matrix, or a set start from 0.

- The expression $\frac{\partial^s \partial^t \boldsymbol{p}(\boldsymbol{x},\boldsymbol{y})}{\partial^s \boldsymbol{x}[i] \partial^t \boldsymbol{y}[j]}$ denotes the sth partial derivative with respect to $\boldsymbol{x}[i]$ and the tth partial derivative with respect to $\boldsymbol{y}[i]$ of the function $\boldsymbol{p}(\boldsymbol{x},\boldsymbol{y})$. For convenience, we also use $f_u(u, v)$ to denote the partial derivative of $f(u, v)$ with respect to u; $f'(u)$ denote derivative of the function $f(u)$ with respect to the variable u.

We also follow several important stylistic rules when assigning variable names. We assign any variable name to be denoted by italics, for example $\boldsymbol{U}$, $\boldsymbol{M}$, a, b, λ. Bold italics denote vectors or matrices, while italics denote scalar variables or names of geometric shapes. Often, a same letter probably has both a bold italic version and an italic version denoting different meanings. For example, M denotes a mesh, while $\boldsymbol{M}$ denotes a matrix.

Roman

Notation of points and vertices does most probably cause confusions. The highlights of these familiar letter notations are the following:

- $\boldsymbol{p}$, $\boldsymbol{q}$, point on continuous curves or surfaces.
- $\boldsymbol{V}$, $\boldsymbol{E}$, $\boldsymbol{F}$, vertex of polygon or mesh or grid. Note that $\boldsymbol{V}$ also denotes a knot vector for spline surfaces;
- $\boldsymbol{v}$, $\boldsymbol{e}$, $\boldsymbol{f}$, vertex of subpolygon or submesh.
- $\vec{\boldsymbol{V}}$, $\vec{\boldsymbol{E}}$, $\vec{\boldsymbol{F}}$, $\vec{\boldsymbol{e}}$, $\vec{\boldsymbol{f}}$, vector formed by vertices.
- $\tilde{\boldsymbol{v}}$, $\tilde{\boldsymbol{e}}$, $\tilde{\boldsymbol{f}}$, vertex after Fourier transformation for $\boldsymbol{v}$, $\boldsymbol{e}$, $\boldsymbol{f}$.

For other some familiar notations, we give their meanings as the following:

- i, j, integer indices
- u, v, continuous parameter variables
- $\boldsymbol{U} = [u_0, u_1, \ldots]$, $\boldsymbol{V} = [v_0, v_1, \ldots]$, knot vector for spline curves or knot vectors for spline surfaces.
- r, s, t, k, l, temporary variables; k usually used as level of subdivision; degree of polynomial; degree of continuity; size of a generation vector. r usually used as multiplicity of a knot u_i in a knot vector $\boldsymbol{U}$ or variable for integer index in a sequence;
- $\boldsymbol{s}(u, v)$ or $\boldsymbol{s}(s, t)$, a part of a subdivision surface
- C^k, G^k, k degree derivative continuity and k degree geometric continuity
- m, n, size of grid, mesh, polygon, matrix, or vector; n also denotes valence of vertex.
- M, M^k, mesh and mesh on kth level of subdivision;
- $N_{i,k}(u)$, $N_i(u)$, B-spline basic function
- d_i, e_i, parameters of vertices or edges in polygons or meshes
- $f()$, $g()$, $p()$, $q()$, $h()$, scalar functions
- d, differential operator
- T, subdivision operator

- x, y, z continuous domain variables. Usually denote coordinates of points or vertices
- $\boldsymbol{s}_i$, $\boldsymbol{t}_i$. generation vector for a grid. It is a unit vector
- D^k, generation vector group formed by generation vectors. It is a set.
- G_D^k, grid formed the vector group D^k
- $\boldsymbol{i} = (i_0,i_1)$, or $\boldsymbol{i} = (i_0,i_1,i_2)$, integer coordinates in a grid
- $\boldsymbol{x}$, $\boldsymbol{y}$, real coordinates in a grid
- $N_{D^k}(\boldsymbol{x})$, box spline basic function
- $C(z)$, generating function for $N_D(\boldsymbol{x})$:
- $X, \overset{s}{X}, \overset{s+}{X}, \overline{\overset{s+}{X}}$, parameter mesh of a manifold patch and sth side of X, extension of $\overset{s}{X}$, rectangular mesh mapped from $\overset{s+}{X}$.
- $c_{i,j}^s$, a chart in the parameter mesh $\overset{s}{X}$,
- w, weight for vertex in a mesh
- T_C, the non-uniform subdivision operator
- T, the skirt-removed operator
- T_{RC}, non-uniform skirt-removed scheme
- T_{CT}, non-uniform subdivision operator constructing subdivision surface interpolating mesh corner vertices
- $C(\bullet)$ denotes the operator taking the continuity degree
- $\boldsymbol{E}$, energy of curve or surface
- $\boldsymbol{M}$, subdivision matrix
- $\boldsymbol{K}$, picking matrices
- S, a usual name for a surface
- $\boldsymbol{S}(\cdot,\cdot)$, $\boldsymbol{S}(\cdot)$ or $\boldsymbol{S}$, a surface equation or any a point on a surface or a mapping from parameter region to a space or a set formed by all points in the surface S.

Greek Letters

α, β, $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$, coefficients or vector formed by coefficients
γ, aperture factor
φ, ϕ, functions or mappings
$\mu(y)$, a mapping constructing the basic curve in parameter regions
κ slope of line
Θ^s A set formed by all related vertices of $\boldsymbol{x}$ in X^s
ε, temporary variables, usually denote very little real number
ζ, a given vector function that represent imposed loads
ξ, eigenvector
Ξ denotes a matrix formed by eigenvectors
Σ, plane
Ω, parameter region

Miscellaneous

D
Z, Z^2, $Z^2/2$, set of integers, set of integer pairs. Set of integer pairs divided by 2
R, R^2, real number space, and two-dimension real number space
$\otimes$, convolution operator of two functions or Kronecker product of matrices
k, curvature

Functions

$supp(D^k)$, support region of a vector group D^k
$O(supp(D^k))$ or $O(n^k)$, an open set which is the inner region of $supp(D^k)$ or a polynomial of n^k
$span(D^k)$, space spanned by D^k
$U(x, \varepsilon)$ a ε neighborhood *of* $\boldsymbol{p}$
$edges(M)$ set of a mesh M
$max\{\bullet\}$, the maximum element of a set denoted by •
$min\{\bullet\}$, the minimum element of a set denoted by •
$|\bullet|$, the valence of a vertex or element number of a set or a vector
$a\%b$, the remainder after a divided by b

Contents

Chapter 1
Introduction

Surface modeling is a fundamental realm of CAD and greatly affects the development of CAD compared with NURBS. Subdivision is a subsequently emerged surface modeling technique. It can be regarded as a bridge between continuous modeling and discrete modeling. Subdivision modeling has a very wide prospect. Just as what DeRose has predicted [1], subdivision will largely supplant B-splines in many application domains in the coming years. This chapter firstly describes the surface modeling. And then, a review for achievements of subdivision surfaces is provided. Lastly, some surveys and books on subdivision modeling are recommended, and meanwhile, the main idea of this book is presented.

1.1 Surface Modeling

Surface modeling generally refers to free-form surface modeling. Different from the analytic surface, the shape of which is determined by an equation, the shape of a free-form surface can vary according to designers' intention. Surface modeling is derived from the aeronautics industry because shapes of airplanes are complex and it is difficult to express these shapes by using analytical surfaces. So far, it has become one of the most important fundamental technologies of CAD/CAM and the most important part of CAGD. It is also the fundamental technologies of many other fields, such as computer graphics, computer animation, computer vision.

Surface modeling mainly researches the representation, design, analysis, and rendering of surfaces in computer graphic systems [2]. These technologies on design, rendering, and analysis depend on representation methods of surfaces. For example, for the point interpolating technique, its implementing algorithms are probably different under different representation methods. Consequently, after a new surface representation appears, people will usually research the technologies of design, analysis, and rendering of the new representation.

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From the middle ages of last century to present, parameter spline method, Coons method, Bézier method, B-spline method, and NURBS method [3–5] have been adopted as main methods of the surface representations. Nowadays, NURBS method is a method that is mostly used. Bézier method and B-spline method can be regarded as special cases of NURBS method. Subdivision method is different from NURBS method. Subdivision surfaces do not have expressions, and they are defined by limits of mesh sequences. T-spline method is another surface representation method that can be regarded as an extension of B-spline method. Due to achievements of operating abilities and memories of PCs, the polygon mesh has become an important shape representation method. A complex shape can be accurately represented by a polygon mesh with hundreds or thousands of polygons.

The polygon mesh is a discrete representation, whereas the NURBS surface and the T-spline surface are continuous representations. A polygon mesh means that their face elements are polygons, for example, triangles, quadrangles, pentagons. The subdivision surface is a representation between the discrete representation and the continuous representation. Because control meshes of spline surfaces and subdivision surfaces are polygon meshes, the polygon mesh is usually mentioned in this book. In this book, *polygon mesh* is simply called as *mesh*. When we use polygon meshes to represent surfaces, especially in the case of meshes with dense vertices, *polygon mesh* is also called as the *mesh surface*. The geometric model expressed by a polygon mesh is called as the *polygon model* or the *mesh model*.

Surface design is a comprehensive topic. We discuss it in two parts: modeling methods and operation methods. Approximation, interpolation, and fitting are three fundamental modeling methods. Figure 1.1 gives explanations for these three modeling methods. There are also other modeling methods, such as skinning, sweeping, extrusion, revolution.

Intersection, trimming, and union are fundamental surface operations that are called Boolean operations. Other surface operations, such as deformation, blending, fairing, reconstruction, simplification, conversion, offset and multiresolution analysis, have been increasingly focused on since the 80s of last century [5, 6].

Surface analysis involves the classical differential geometric properties of surfaces, such as continuity, curvature, principle directions. Surface analysis estimates

(a) Approximation; (b) Interpolation; (c) Fitting

Fig. 1.1 Three fundamental modeling methods the qualities of surfaces and abilities of modeling methods. It can guide modifications of surface shapes and constructions of modeling methods.

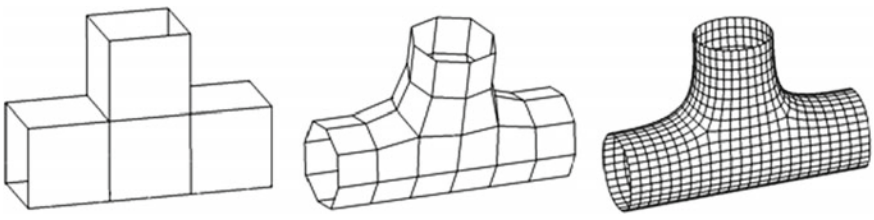
Surface rendering involves drawing, color, light, texture, *etc.*. Hardware rendering is an important research area of subdivision surfaces. In order to rapidly render subdivision surfaces, an efficient method is to subdivide control meshes using hardware.

1.2 Concept of Subdivision Surfaces

Although NURBS has become the industry standard of data exchange of geometric information in computers and is widely applied in industry and animation modeling, it still has some disadvantages. Just as what DeRose [7] has pointed out, a single NURBS surface cannot express a complex surface with arbitrary topology, for example, surfaces in human animation modeling. Patchwork of trimmed NURBS is the most commonly adopted way to model complex smooth surfaces. However, this way does have at least two difficulties: (1) Trimming is expensive and prone to numerical error; (2) it is difficult to maintain smoothness, or even approximate smoothness, at the seams of the patchwork as the model is animated.

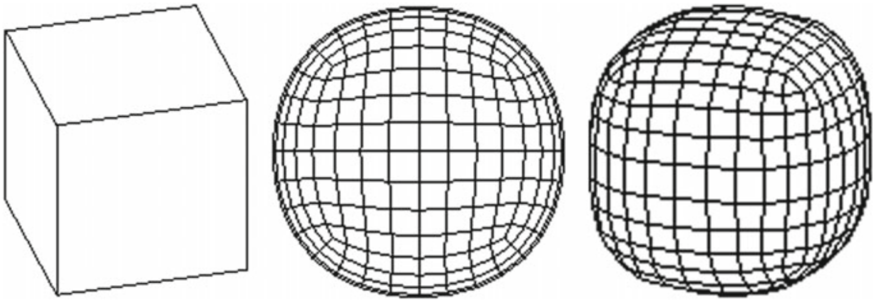
Subdivision surfaces have the potential to overcome these problems: They do not require trimming, and smoothness of the model is automatically guaranteed, even as the model animates. They have advantages of mesh surfaces and spline surfaces.

What is subdivision? Subdivision densifies vertices of meshes by a set of given rules. The process of densifying vertices of meshes is also called refinement. A mesh sequence is obtained when we recursively subdivide a mesh. The limit of the mesh sequence is called a subdivision surface if the mesh sequence is convergent. Any mesh in the sequence is called a subdivision mesh. If a mesh has the same topology as that of a subdivision mesh, we call that the mesh has subdivision connectivity. A set of subdivision rules is called a subdivision scheme. In applications, we usually replace the subdivision surface by using the subdivision mesh on a certain subdivision level. Figure 1.2 gives a construction process of a subdivision surface. In this case, the pipeline is expressed by an entire surface without trimmings and joins.



(a) Initial mesh; (b) Subdivide once; (c) Subdivide twice

Fig. 1.2 Construction process of a subdivision surface



(a) Initial mesh; (b) C-C subdivision scheme; (c) Doo-Sabin subdivision scheme

Fig. 1.3 Different subdivision schemes produce different subdivision surfaces for the same initial mesh

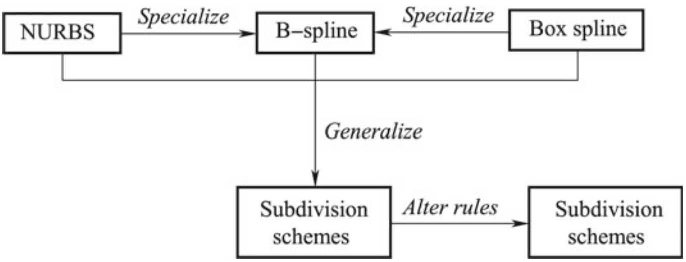


Fig. 1.4 Relations between splines and subdivision schemes

Subdivision is closely related to box spline theories. How is it related to box splines? A box spline surface has a control mesh. For convenience, we call the subdivision of control meshes of spline surfaces as refinement. Box spline theories define refinement rules for control meshes. The limit surface is the box spline surface when we refine a control mesh. We obtain a subdivision scheme if we generalize these refinement rules to arbitrary meshes. It is one of the subdivision scheme construction methods to generalize refinement rules to arbitrary topology meshes. Different subdivision schemes can be obtained based on different types of box spline surfaces. For the same mesh, different subdivision surfaces can be obtained by using different subdivision schemes, which is shown in Fig. 1.3. Another subdivision scheme will be produced if we alter some of the subdivision rules of a subdivision scheme, which is regarded as another method to construct new subdivision schemes. Relations between splines and subdivision schemes are shown in Fig. 1.4. From the figure, it should be noticed that B-spline is a special case of box splines.

As a generalization of spline surfaces, subdivision surfaces can be regarded as bridges between continuous surfaces and discrete surfaces. On the one hand, we can replace the subdivision surface by using the subdivision mesh on a certain subdivision level. On the other hand, we can analyze geometric properties of subdivision surfaces in the view of continuous surfaces. Generally, subdivision elegantly addresses many issues that are confronted in computer graphics [8]:

Arbitrary Topology: Subdivision generalizes classical spline surface approaches to arbitrary topology. This implies that there is no need for trimming curves or awkward constraint management between patches.
Scalability: Because of its recursive structure, subdivision naturally accommodates level-of-detail rendering and adaptive approximation with error bounds. The result let us be able to make the best use of limited hardware resources, such as those found on low-end PCs.
Uniformity of Representation: Many of traditional modeling methods use either polygon meshes or spline patches. Subdivision spans the spectrum between these two extremes. Surfaces can behave as if they are made of patches, or they can be treated as if consisting of many small polygons.
Numerical Stability: The meshes produced by subdivision have many of the nice properties finite element solvers require. As a result, subdivision representations are also highly suitable for many numerical simulation tasks which are important in engineering and computer animation settings.
Code Simplicity: Subdivision is simple to implement and very efficient to execute.

1.3 Development of Subdivision Surfaces

Though subdivision surfaces have prevailed since the late 1990s, the basic ideas behind subdivision are very old indeed and can be traced as far back as the early 1950s when De Rham G. used "corner cutting" to describe smooth curves. However, the application in geometric modeling starts with the proposal of Chaikin [9], who devised a method of generating smooth curves for plotting in the middle 1970s. In the limit, Chaikin's algorithm produced uniform quadratic B-spline curves. In 1978, Catmull and Clark [10], and Doo and Sabin [11] generalized refinement rules of control meshes of biquadratic and bicubic B-spline surfaces to meshes of arbitrary topology, which indicates that subdivision formally becomes one of the surface modeling methods. The generalization of biquadratic B-spline surfaces is called Doo–Sabin subdivision surfaces, and their subdivision rules are called Doo–Sabin subdivision scheme. Similarly, we have Catmull–Clark subdivision scheme that is usually simply called as C-C subdivision scheme.

The number of mesh vertices exponentially increases when a mesh is subdivided. For example, there are totally m vertices on a mesh. If the mesh is subdivided k times and a vertex becomes n vertices after a subdivision step, there are approximately mn^k vertices on the mesh. Limited to computers' abilities of operation and memories, subdivision has not been focused on until the 90s of last century. Before the 90s of last century, most researchers mainly have paid attention to NURBS surfaces. The theoretical system of NURBS became mature. However, subdivision techniques also constantly grew in this period. The growth was embedded in two aspects. One is construction of subdivision schemes and the other is analysis of subdivision surfaces.

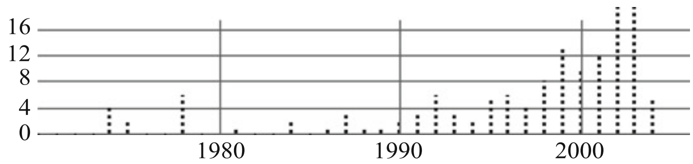


Fig. 1.5 Subdivision papers, plotted by year

For construction of subdivision schemes, there are also two big ideas. One is called Loop subdivision scheme and the other is the four-point scheme. Loop subdivision scheme was described in Loop's Masters thesis [12] in 1987, and it was defined over a grid of triangles. The four-point scheme was presented by Dyn, Levin, and Gregory [13] in 1987. It was a subdivision scheme of curves. That is, we subdivide a polygon and then obtain a curve by using the subdivision scheme. Different from the previous approximating subdivision schemes, it was an interpolating scheme. The generalization of the four-point scheme began in 1990 [14]. The subdivision scheme is called Butterfly scheme that is also defined over a triangular grid.

For the analysis of subdivision surfaces, we pay attention to convergence of subdivision mesh sequences and continuity of subdivision surfaces. For simplicity, we call them convergence and continuity of subdivision schemes, respectively. In this period, the main method of subdivision surface analysis is the eigenanalysis of subdivision matrices. In 1978, Doo and Sabin [11] gave the famous eigenanalysis method—discrete Fourier transformation. In later eigenanalysis analysis, Fourier transformation was always used without exception. As the development and application of Doo and Sabin's idea, Ball and Story considered a more general form of the algorithm of Doo and Sabin [15]. Loop also gave eigenanalysis of his Loop subdivision surfaces when constructing the subdivision scheme. Ball et al.'s work and Loop's work show that the eigenanalysis of subdivision matrices could be used explicitly in the original design of a scheme.

After 1990, especially in the late 1990s and early 2000s, subdivision modeling techniques have developed rapidly. The opinion is reflected by the statistics in [16]. The statistics is given in Fig. 1.5. We summarize these achievements in the following ten aspects:

(1) Subdivision surface analysis. A representative work was given by Reif [17] in 1995. He identified that we could not ensure that the subdivision surfaces were G^1 continuous only by using properties of subdivision matrices. Consequently, he constructed the characteristic map and gave sufficient and necessary conditions for G^1 continuity of subdivision surfaces. By using the theory, Reif [18] showed that the attempts to make a G^2 variant without flat points for a C-C surface will not succeed, which evoked many discussions on how to construct G^2 continuous subdivision surfaces in later years. General criteria to construct G^k continuous subdivision surfaces were given by Zorin in his Phd thesis [19] in 1998. By using these criteria, he designed an algorithm to verify G^1 continuity of subdivision surfaces. Prautzsch [20] generalizes results in Reif [17] and gave the sufficient and necessary conditions

for G^k continuity of subdivision surfaces. As a shape analysis method, Peter [21, 22] considered the differential geometric properties of subdivision surfaces, such as the fundamental forms, the Weingarten map, the principal curvatures, the principal directions in 2004. By relating the shape properties to the subdivision matrices, they obtained some conditions for the construction of high-quality subdivision schemes.

(2) New subdivision schemes. Methods to construct subdivision schemes are classified into two types: One is to design new subdivision schemes, and the other is to alter old subdivision schemes to improve continuity of subdivision surfaces, especially to obtain G^2 continuous subdivision surfaces, in light of subdivision surface analysis. For the first type, Kobbelt [23] presented an interpolating subdivision scheme over quadrilateral grids. The scheme is a generalization of product of the four-point scheme for curve subdivisions. Peters and Reif [24] presented the "simplest" scheme that was also midpoint subdivision scheme. The midpoint subdivision surface was in fact a generalization of a box spline surface. Sederberg [25] gave the non-uniform subdivision schemes by generalizing non-uniform quadratic and cubic B-spline surfaces. Kobbelt [26] described a $\sqrt{3}$ subdivision scheme defined over a grid of triangles. Compared with the Loop scheme, the number of faces of subdivision meshes produced by a $\sqrt{3}$ subdivision scheme increases slowly. When a subdivision step is executed, the number of faces has an increase of a $\sqrt{3}$ multiple. The theme on the increase rate leads to Velho's 4–8 scheme [27]. Based on 4–8 scheme, Li [28] gave a $\sqrt{2}$ subdivision scheme. Dyn [29] introduced a hexagon subdivision scheme that constructed interpolating convexity-preserving subdivision surfaces. Different from Dyn's hexagon subdivision scheme, Zhang's hexagon subdivision scheme [30] was an approximating scheme. For the second type, Zorin [8, 31] gave new subdivision rules for Butterfly subdivision scheme. The improved Butterfly subdivision scheme can construct C^1 continuous subdivision surfaces. By using the sufficient and necessary conditions for G^k continuity of subdivision surfaces, Prautzsch and Umlauf [32–34] altered the subdivision rules of C-C subdivision scheme, Loop subdivision scheme, and Butterfly subdivision scheme. After altering the subdivision rules, C-C subdivision surfaces and Loop subdivision surfaces are G^2 continuous and Butterfly subdivision surfaces are G^1 continuous. It is a character of the period that new subdivision schemes bloomed up, which was due to two reasons. One is that the relations between box splines and subdivision are opened out. Because every box spline has a set of refinement rules, we can generalize these refinement rules to arbitrary meshes and obtain new subdivision schemes. Of course, each such scheme would have to have its extraordinary point rules invented. The other is that the construction of subdivision schemes has the directions of subdivision surface analysis. Subdivision surface analysis is helpful to invent extraordinary point rules of subdivision schemes.

(3) Classifications of subdivision schemes. It is an important task to classify these subdivision schemes since there are so many subdivision schemes. In fact, we have used some classification methods in above discussions, such as interpolating subdivision and approximating subdivision, Zorin et al. [8] enumerated some basic classification methods. The topic was profoundly researched after 2000. Ivrissimtzis et al. [35] gave a classification system for subdivision schemes by using similarity

transformations of grids. The classification is a generalization of Alexa' classification [36]. The classification shows that subdivision schemes with low incensement ratio of mesh elements (vertices, edges, and faces) come at the expense of symmetry and uniformity. It is a natural idea to relate classification and unification of subdivision schemes. Stam [37] presented a class of subdivision surfaces which generalized uniform tensor product B-spline surfaces of any bi-degree to meshes of arbitrary topology. In the class, Doo–Sabin subdivision scheme and C-C subdivision scheme become two special cases. Zorin [38] gave an analogous work almost at the same time. Similar to the generalization of B-spline surfaces of any bi-degree, Oswald [39] presented a new family for a$\sqrt{3}$ subdivision. After inserting new vertices into meshes, there are smooth iterative steps. When iterative times increase, the resulting surfaces become smoother at regular vertices. As a further generalization of the new family, they also introduced a wider class of composite subdivision schemes suitable for arbitrary topologies and topology rules of subdivisions.

(4) Parameter evaluation. After taking a pair of parameters (u, v), can we compute the coordinates of $\boldsymbol{S}(u, v)$? Parameter evaluation can answer this question. Except for coordinates of points, parameter evaluation methods can usually compute partial derivatives of subdivision surfaces. Consequently, we can compute some differential geometric variables, such as normals, curvature, and principle directions, if these variables exist. Stam's method [40] is perfect in theory, and it can evaluate subdivision surfaces generalized from spline surfaces [41]. Based on Stam's method, Wang [42] gave a parameter evaluation method for non-uniform C-C subdivision scheme. Different from Stam's method, Lai [43] gave another evaluation method that employed less eigenbasis functions. Halstead et al. [44] gave formulas to compute limit positions and limit normals of mesh vertices of subdivision surfaces. Though their methods cannot be regarded as parameter evaluation methods, those formulas are convenient to compute limit properties of mesh vertices.

(5) Interpolation. The interpolating methods based on subdivision surfaces can be classified into two categories: interpolation of vertices and interpolation of curves. The interpolation of vertices falls into two topics: interpolating subdivision and approximating subdivision. The famous interpolating subdivision scheme—Butterfly scheme appeared in 1990. Two sets of improved rules of Butterfly scheme were, respectively, given by Zorin [8, 31] and Prautzsch and Umlauf [20, 32]. Kobbelt's interpolating subdivision scheme was a generalization of the four-point scheme and was given in 1996 [23]. Except for these schemes generalized from the four-point scheme, there are some schemes obtained by altering approximating schemes. Labisk and Greiner [45] gave an interpolating $\sqrt{3}$ subdivision scheme. Li et al. [46] gave an interpolating $\sqrt{2}$ subdivision scheme. Zhang [47] gave push-back interpolating subdivision schemes for C-C subdivision scheme, Doo–Sabin subdivision scheme, and Loop subdivision scheme. For algorithms based on approximating subdivision schemes, Nasri [48] noticed that Doo–Sabin subdivision surfaces interpolate the center of each face of meshes. Consequently, control vertices can be computed by linear systems. Based on the idea, Nasri [49] presented a method to construct Doo–Sabin subdivision surfaces interpolating given points and normals. Similar to Nasri's method, Li et al. [50] discussed the method of interpolation of points and

normals based on C-C subdivision scheme. Zheng et al.'s method [51] was similar to Li's method while Zheng considered the fairing of C-C subdivisions. In order to improve the fairing of subdivision surfaces, Zheng adjusted control vertices by using local optimization models. By using shape similarity, Lai [52] also gave an interpolating method of C-C subdivision surfaces. Helatead's fairing interpolating method [44] was very representative, and he used physics energy model to compute control vertices. Helatead's method was a global method. It was an interesting phenomenon that interpolations of points are always related to interpolations of normals when people construct interpolating surfaces using approximating subdivisions. For the interpolation of curves, the representative algorithm was Levin's combined subdivision schemes [53] and the algorithm was also applied in trimming of subdivision surfaces, filling of holes, and blending of surfaces [54–56]. By using polygonal complexes, Nasri [57] researched the curve interpolations of Doo–Sabin subdivision surfaces and C-C subdivision surfaces. Zhang et al.'s method [58] can be regarded as developments of the polygonal complex method. In Zhang's method [58], polygonal complexes are called "symmetric zonal meshes."

(6) Fitting. Fitting is a key of reverse engineering. It is usually called the surface reconstruction that fits unstructured triangle meshes or points clouds by using subdivision surfaces. Fitting can mainly be classified into two categories: local parameterization method [59] and circularly adjusting-vertices method [60, 61]. The first method constructs harmonic maps between a coarse polygon model and the fitted original surface. The process of constructing harmonic maps is also the process of parameterization. By using harmonic maps, we sample the original surfaces. These samples form a mesh with subdivision connectivity. That is, these meshes are not obtained by subdivisions while they have topologies of subdivision meshes. The second method adjusts vertices of meshes after each subdivision step in order that the shape of subdivision meshes or limit surfaces is as close as possible to the shape of original surfaces. Ma's method [62] does not fall into the above two categories. After parameterizing the original surface, he computes control vertices by the least square method. Since shapes in the real world are not all smooth and they probably have some sharp characters, such as creases, darts, sharp points, some special subdivision rules have to be designed for existing subdivision schemes in order to truly fit these sharp characters [63].

(7) Multiresolution analysis of subdivision surfaces. For a mesh M^k with subdivision connectivity, how should another mesh M^{k-1} with the subdivision connectivity in a lower subdivision level be chosen to approximate the shape? How do we restore the shape of M^k from M^{k-1}? Multiresolution analysis of subdivision surfaces asks the question. Wavelet is an important tool of multiresolution analysis of subdivision surfaces. Wavelet transformation of subdivision surfaces is originally explored by Lounsbery et al. [64]. Based on lifting scheme for B-spline wavelets, Martin [65, 66] presented a construction method of lifted biorthogonal wavelets for meshes with C-C subdivision connectivity. Their algorithm can be executed in linear time. Not using wavelet transformations, Zorin [67] described a multiresolution representation for meshes based on subdivision. Based on the representation, they built a scalable interactive multiresolution editing system.

(8) Boolean operations. Intersection and trimming of surfaces are the basis of Boolean operations. Litke [54] researched the trimming of subdivision surfaces by using the combined subdivision scheme. The combined subdivision scheme ensured the surfaces after trimming accurately interpolate trimming curves. Using the union operation as an example, Biermann [68] discussed the Boolean operation of subdivision surfaces defined on triangle meshes. In Biermann's method, it is important to construct multiresolution meshes for resulting surfaces of union operations. Hui [69] presented an algorithm for blending of subdivision surfaces. Zhou [70] also researched the intersection and trimming operations of Loop subdivision surfaces. Generally, it is the key of Boolean operations to control the precision of operation results.

(9) Rendering. How do we rapidly render a subdivision surface? This is a problem on evaluation of subdivision surfaces [71]. The evaluation methods of subdivision surfaces can be classified into two categories: software evaluation and hardware evaluation. Recursive subdivision is the most direct software evaluation method. Since ordinary recursive subdivision results in exponential increase of elements of meshes, adaptive subdivisions are considered by many researchers. Different from ordinary recursive subdivisions, adaptive subdivision results in such resulting meshes: There are higher subdivision levels on regions with larger curvatures. Kobbelt [23] is one of the researchers that first used the adaptive subdivision. The adaptive subdivision scheme is given for quadrilateral subdivisions. The Y-split technique in the adaptive subdivision is a famous tackle to repair holes between regions with different subdivision levels. Kobbelt [26] also presented an adaptive subdivision method for a $\sqrt{3}$ subdivision scheme. The adaptive subdivision method still focused on eliminating gaps between regions with different subdivision levels. Most literature on adaptive subdivisions appeared in the 2000s and late of 1990s. Li [72] presented a survey for adaptive subdivisions. He considered that there were three topics for adaptive subdivisions: criteria to determine regions that should have high subdivision levels, methods to eliminate gaps between regions with different subdivision levels, and rules to compute coordinates of vertices. By combining C-C subdivision with T-spline method, Sederberg [73] presented T-NURCC subdivision scheme that can also be regarded as an adaptive subdivision scheme. The topic on the hardware evaluation focuses on balancing the workload between CPU and GPU (graphics processing unit). These algorithms take advantage of parallel execution streams in programmable graphics hardware. This is an interesting topic after 2000. In 2000, Bischoff et al. [74] proposed a hardware solution for Loop subdivision surface rendering. The pretabulated basis function composition method explored by Bolz [71] is a representative hardware rendering method for C-C subdivision surfaces. Unfortunately, the tabulated evaluation limits flexibility and can increase downstream complexity, and it is probably troublesome to produce some sharp characters. By contrast, Shiue's GPU subdivision kernel [75] generated the subdivision mesh at different levels on the GPU so that all evaluation work rested with GPU shaders.

(10) Applications. It is a milestone that the animation short film *Geri's game* succeeded in 1998. Its dramatis personae model is made by using subdivision techniques [7]. So far, some leading animation software, such as Maya, 3D-Max, has used the

subdivision as a main modeling method. The 16^{th} part of the MPEG4 standard—DAFX(Animation Framework extension)—also introduces the subdivision as a main modeling method. The first AFX version was released in the beginning of 2003. In 2006, a new AFX version was released.

Generally, every topic about surface modeling is almost referred in the researches of subdivision surfaces after 2000. Except for the above topics, there are also some other topics, such as mixed subdivision [16, 70], surface deformation [76], surface conversion [77, 78], surface offset [79–81].

1.4 Idea of This Book

So far, subdivision surface modeling techniques have formed a perfect system from theories to applications. Furthermore, applications of subdivision surfaces have reached great success in the animation field. Subdivision surfaces have become data exchange standards of 3D animations. Practices show that the subdivision is a powerful modeling tool and has the powerful potential. Consequently, it is a significant task to sum up existing researching results and generalize them. However, these achievements on subdivision surfaces are abundant and profound. It is impossible for this book to involve all these achievements. Consequently, this book aims at gathering achievements in aspect of applications. Some contents that the authors think are difficult, such as conditions for G^k continuity [19, 20, 32–34], variational subdivision [82, 83], Loudbery's wavelet transformation [64], hardware rendering [71, 74, 75], will not be introduced in this book. We hope that this book can lead beginners to the subdivision modeling field.

This book is not the first work that summarizes achievements of subdivision modeling. There have been some surveys and books on subdivision before this book. Literatures [8, 16, 86] are all-around surveys. The survey [8] summarizes achievements before 1998 based on the following topics: relations between subdivision and B-splines, analysis of convergence and continuity of subdivision surfaces. Discussions of the survey [8] are very detailed and can be considered as explanations for original literatures. The survey [86] summarizes achievements of subdivision surfaces before 2004. The survey puts emphasis on relations between subdivisions and refinements of B-splines. Some common issues on subdivision surface modeling are also addressed. Several key topics, such as subdivision scheme construction, subdivision surface property analysis, parametric evaluation, and subdivision surface fitting, are discussed. Some other important topics are also summarized for potential future research and development. Compared with the above surveys, The survey [16] contains the most comprehensive contents. He summarized achievements before 2003 and reviewed those new ideas and new methods after 1995. Except for reviews of new subdivision schemes and classification methods, he still referred to multivariate subdivision and face-valued subdivision and discusses non-uniform subdivision, non-stationary subdivision, mixed subdivision etc., as new ideas. For subdivision surface analysis, shape tuning and the parameter evaluation, he gives advantages and disadvantages of some main methods. For applications of subdivision surfaces, he discusses finite element, data compression, and reverse engineering. The survey [87] gathers these achievements of subdivision surfaces before 2006 in two parts: fundamental theories and application techniques. Compared with the above surveys, he still discusses hardware rendering techniques and applications in animations.

The literature [1] is a monograph on subdivision surfaces and discusses achievements before 2001. This book mainly discusses the fundamental theories of subdivisions. It points out that subdivision may be viewed as the synthesis of two previously distinct approaches to modeling shape: functions and fractals. Construction of subdivision schemes is a main content of the book. Analysis of convergence and continuity of subdivision surfaces are also discussed as a "hot" topic in that period. There are other some monographs [6, 88, 89] on surface modeling that introduce some knowledge of subdivision surfaces. However, only several classical subdivision schemes and basic principles of subdivision surface analysis are discussed in those books.

The outline of this book is designed according to the idea of [87] and consists of roughly two parts: fundamental theories and applied techniques. The first part includes relations between subdivision and splines, introductions of some main subdivision schemes, and theories of subdivision surface analysis. Different from other surveys and books, this book directly starts from the recursive definition of B-spline and hopes that give readers a concise cognition of spline theories. The method to construct subdivision schemes from box splines is a main content of this part. The second part includes n-side patches, optimization modeling interpolation, fitting, deformations, intersection and trimming, mesh editing. Those technologies are necessary to construct geometric models using subdivision surfaces.

Remarks

This chapter gives a review for the development of subdivision surfaces. This review focuses on the stationary subdivision scheme though there are also some literatures on the non-stationary subdivision [82, 83, 90]. Stationary subdivision schemes currently are the most frequently applied subdivision schemes. Discussions in this book will also focus on stationary subdivision schemes. As far as expressions of geometric shapes are concerned, there are three forms: univariate, bivariate, and trivariate which are, respectively, fits for curves, surfaces, and volumes. There are probably expressions of higher dimensions [16]. However, our discussions are also limited to bivariate subdivision schemes, i.e., subdivision schemes to construct surfaces. Surfaces are the most frequently applied geometric shapes. A majority of researches on subdivision modeling are researches of subdivision surfaces. Topics involved by researches of subdivision surfaces are very extensive. These topics discussed in this book are only several ones. However, we try to enumerate other topics and literatures so that the readers can know them. For meshes, there are manifold meshes and nonmanifold meshes. This book also refers to two-manifold meshes because a single surface is a two-manifold in the differential geometric.

In this chapter, there are probably some concepts that are not known by beginners. We have given simple descriptions for some elementary concepts such as **mesh, subdivision scheme, triangle subdivision**. Strict definitions will be given in later chapters. It is a good idea for a beginner to directly read the second chapter if he or she does not have a clear understanding of those elementary concepts. After reading the second chapter, you may read the introduction again. We want to give readers a comprehensive cognition for subdivision surfaces by the introduction.

Note: In some literatures, C^k continuity of subdivision surfaces is discussed, while G^k continuity of subdivision surfaces is discussed in other literatures. In the view of derivations, C^k continuity and G^k continuity are different. That is to say, C^k continuity is a concept related to expressions of curves and surfaces, while G^k continuity is related to geometric variables. However, it is easy to know that a surface must be G^k continuous if it is C^k continuous. Consequently, for geometric shapes, we do not use the concept of G^k continuity except special requirements. For functions whose values are scalar, we do use the concept of C^k continuity.

Exercises

1. What is the polygon mesh? Why is a polygon mesh able to represent a complex shape?
2. What is the subdivision surface? Why is a single subdivision surface able to represent a complex shape but why a single NURBS surface hasn't the ability?
3. Why do most of the polygons of a mesh have the similar shape after the mesh is subdivided several times?

Chapter 2
Splines and Subdivision

Most subdivision schemes are derived from refinement methods of control meshes of spline curves and surfaces. This chapter mainly discusses various refinement methods for control meshes of spline curves and surfaces. Firstly, definitions and basic properties of spline functions are introduced; secondly, refinement rules of spline functions are deduced based on their definitions and basic properties; lastly, refinement rules of control meshes of spline curves and surfaces are deduced based on refinement rules of spline functions. In next chapter, we will generalize these refinement rules of control meshes to arbitrary 2-manifold meshes to construct subdivision surfaces.

2.1 B-Splines

There are many definitions for B-splines. The recursive definition is the most commonly used one in computational field. Its discovery is attributed to de-Boor, Cox, and Mansfield [3, 4].

Definition 2.1 Given a non-decreasing sequence of the real u-axis:

$$\boldsymbol{U} = [\cdots, u_{-2}, u_{-1}, u_0, u_1, u_2, \cdots], \text{ where } u_i \leqslant u_{i+1}.$$

B-splines (or B-spline basic functions) can be defined as:

$$\begin{cases} N_{i,0}(u) = \begin{cases} 1, & if\ u_i \leqslant u < u_{i+1} \\ 0, & otherwise \end{cases} \\ N_{i,k}(u) = \dfrac{u - u_i}{u_{i+k} - u_i} N_{i,k-1}(u) + \dfrac{u_{i+k+1} - u}{u_{i+k+1} - u_{i+1}} N_{i+1,k-1}(u) \\ assume\ 0/0 = 0. \end{cases} \tag{2.1}$$

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