Visualization and Processing of Tensor Fields

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Preface

Matrix-valued data sets – so-called tensor fields – have gained significant importance in the fields of scientific visualization and image processing. The tensor concept is a common physical description of anisotropic behaviour, especially in solid mechanics and civil engineering. It arises e.g. in the measurements of stress and strain, inertia, permeability and diffusion. In the field of medical imaging, diffusion tensor magnetic resonance imaging (DT-MRI) has become widespread in order to gain valuable insights into connectivity properties of the brain. Tensors have also shown their use as feature descriptors in image analysis, segmentation and grouping.

These recent developments have created the need for appropriate tools for visualizing and processing tensor fields. Due to the multivariate structure of the data and their multidimensional variation in space, tensor field visualization belongs to the most challenging topics in the area of scientific visualization. Moreover, most signal and image processing methods have been developed for scalar- and vector-valued data sets, and only recently researchers have tried to investigate how they can be extended to tensor fields. In this case one has to take into account a number of additional constraints such as an appropriate coupling of the different channels, and preservation of properties such as positive semidefiniteness during the filtering process.

Unfortunately the results in the field of tensor-valued visualization and image processing are scattered in the literature, and often researchers in one application area are not aware of recent progress in another area. In order to address this problem, the editors of this book organized a perspective workshop that took place at Schloss Dagstuhl, Germany from April 18 to 23, 2004. In that week 30 invited scientists, representing many of the world-wide leading experts in tensor field visualization and processing, had the opportunity to exchange ideas in a highly inspiring atmosphere. Since many of them met for the first time, this exchange proved to be particularly fruitful.

One of these fruits was the wish of all participants to compile their knowledge in a single edited volume. The present book – which is the first of its kind in this field – is the result of these efforts. Its goal is to present the state-of-the-art in the visualization and processing of tensor fields, both as an overview for the inquiring scientist, and as a basic foundation for developers and practitioners. The book contains some longer chapters dedicated to surveys and tutorials of specific topics, as well as a great deal of original work not previously published. In all cases the emphasis has been on presenting the details necessary for others to reproduce the techniques and algorithms. Another goal of this book is to provide the basic material for teaching state-of-the-art techniques in tensor field visualization and processing. It can therefore also serve as a textbook for specialized classes and seminars for graduate and doctoral students. An extended bibliography is included at the end of each chapter pointing out where to obtain further information.

Organization of the Book

This volume consists of 25 chapters. Each of them has been reviewed by two experts and carefully revised according to their suggestions. The chapters are arranged in five thematic areas. Color plates can be found in the Appendix.

The book starts with an introductory chapter by Hagen and Garth. It gives the mathematical background from linear algebra and differential geometry that is necessary for understanding the concept of tensor fields.

Part I of the book is devoted to feature detection using tensors. Here one typically starts with scalar- or vector-valued images and creates tensor-valued features that are suitable for corner detection, texture analysis or optic flow estimation. Structure tensors are the most prominent representatives of these concepts. Chapter 2 by Brox et al. is a survey chapter on adaptive structure tensors, while Chap. 3 by Nagel analyzes closed form solutions for a structure tensor concept in image sequence analysis. An alternative framework for tensor-valued feature detection is presented in Chap. 4 by Köthe who shows that the so-called boundary tensor may overcome some problems of more traditional approaches.

Part II deals with the currently most important technique for creating tensor-valued images, namely Diffusion Tensor Magnetic Resonance Imaging (DT-MRI), often simply called Diffusion Tensor Imaging (DTI). This technique measures the diffusion properties of water molecules and has gained significant popularity in medical imaging of the brain. Chapter 5 by Alexander gives a general overview of the principles of biomedical diffusion MRI and algorithms for reconstructing the diffusion tensor fields, while the subsequent chapter by Hahn et al. describes the empirical origin of noise and analyzes its influence on the DT-MRI variables. After having understood the formation of DT-MR images their adequate visualization remains a challenging task. This is the topic of the survey chapter by Vilanova et al. that also sketches the clinical impact of DT-MR imaging. A more specific medical application is treated in Chap. 8 by Gee et al. who study anatomical labeling of cerebral white matter in DT-MR images. For conventional DT-MR imaging, the identification and analysis of fibre crossings constitutes a severe difficulty. In Chap. 9, Pasternak et al. introduce a variational image processing framework for resolving these ambiguities. An alternative strategy is investigated in Chap. 10 by Özarslan et al. They reconstruct higher-order tensors from the MR measurements. These allow to encode a richer orientational heterogeneity.

In the third part, general visualization strategies for tensor fields are explored. This part starts with a review chapter by Benger and Hege who also consider applications in relativity theory. Kindlmann's chapter is concerned with visualizing discontinuities in tensor fields by computing the gradients of invariants. The subsequent Chaps. 13–16 investigate different strategies for understanding the complex nature of tensor fields by extracting suitable differential geometric information. While Chap. 13 by Tricoche et al. deals with the topology and simplification of static and time-variant 2-D tensor fields, Chap. 14 by Zheng et al. is concerned with a novel and numerically stable analysis of degerated tensors in 3-D fields. In Chap. 15, Wischgoll and Meyer investigate the detection of alternative topological features, namely closed hyperstreamlines. The third part is concluded with a chapter by Hotz et al. who introduce specific visualization concepts for stress and strain tensor fields by interpreting them as distortions of a flat metric.

Part IV of the book is concerned with transformations of tensor fields, in particular interpolation and registration strategies. In Chap. 17, Moakher and Batchelor perform a differential geometric analysis of the space of positive definite tensors in order to derive appropriate interpolation methods. The next chapter by Pajevic et al. deals with non-uniform rational B-splines (NURBS) as a flexible interpolation tool, while in Chap. 19 Weickert and Welk introduce a rotationally invariant framework for tensor field approximation, interpolation and inpainting. It is based on partial differential equations (PDEs). Finally, Chap. 20 by Gee and Alexander treats the problem of diffusion tensor registration. Compared to scalar-valued registration approaches, incorporating the orientation information provides additional challenges.

The fifth part is a collection of contributions on signal and image processing methods that are specifically developed to deal with tensor fields. Chapter 21 by Welk et al. as well as Chap. 22 by Burgeth et al. show that seemingly simple ideas like median filtering and morphological image processing can create substantial difficulties when one wants to generalize them to tensor fields. Since tensor lack a full ordering, many straightforward concepts cannot be applied and alternative generalizations become necessary. In Chap. 23, Suárez-Santana et al. review adaptive local filters for tensor field regularization and interpolation that are steered by a structure tensor, while Chap. 24 by Westin et al. is concerned with two other regularization techniques for tensor fields: normalized convolution and Markov random fields. These ideas a complemented by Chap. 25 where Weickert et al. survey the most important PDE approaches for discontinuity-preserving smoothing and segmentation of tensor fields.

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It is a pleasure to thank all authors for their excellent contributions. We are very grateful to Reinhard Wilhelm and the Scientific Directorate for accepting our proposal for a Dagstuhl Perspective Workshop on Visualization and Processing of Tensor Fields, and the local team at Schloss Dagstuhl for providing a very pleasant atmosphere that allowed all participants to enjoy this workshop with real scientific benefits. Moreover, we thank the editors of the Springer book series *Mathematics and Visualization* as well as Martin Peters and Ute McCrory (Springer, Heidelberg) for their support to publish this edited volume in their series. Last but not least, we are greatly indebted to Martin Welk (Saarland University) who was of substantial help in the final compilation of all chapters to a coherent, single volume.

Saarbrücken and Kaiserslautern, June 2005 — *Joachim Weickert*, *Hans Hagen*

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Introduction

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An Introduction to Tensors

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Summary. This chapter is a short introduction into tensor fields, some basic techniques from linear algebra, differential geometry and the mathematical concept of tensor fields are presented. The main goal of this chapter is to give readers from different backgrounds some fundamentals to access the research papers in the following chapters.

1.1 Some Linear Algebra

Remark: Since we are only able to sketch out some of the basic facts of linear algebra, the reader is referred to a comprehensive body of literature on the topic. For example, the book by Fuhrmann [1] provides an introduction in a modern language.

1.1.1 Bases and Basis Transforms

Let U a vector space over a field $\mathbb{F}$ (e.g. $\mathbb{F} = \mathbb{R}$ or $\mathbb{F} = \mathbb{C}$). A set of elements $\{a_1, \ldots, a_n\} \subset U$, $n \in \mathbb{N}$, is called a basis if every $u \in U$ admits a unique non-trivial linear combination of the a_i over $\mathbb{F}$:

$$u = u_1 a_1 + \cdots + u_n a_n$$

The coefficients u_i give rise to the vector notation of $u \in U$ with respect to this basis as

$$u = \begin{pmatrix} u_1 \\ \vdots \\ u_n \end{pmatrix}.$$

We say that U has dimension n, since every basis has exactly n elements and every $u \in U$ can be described by n elements of $\mathbb{F}$ (coordinates).

Let V a vector space over $\mathbb{F}$ with basis $(b_1, \ldots, b_m)$, $m \in \mathbb{N}$. Hence, $\dim V = m$. A linear map $L : U \to V$ is a map satisfying

$$L(\alpha u + \beta u') = \alpha L(u) + \beta L(u') \qquad \forall u, u' \in U, \forall \alpha, \beta \in \mathbb{F} \ .$$

We can observe

$$L(u) = L\left(\sum_{i=1}^{n} u_i a_i\right) = \sum_{i=1}^{n} u_i L(a_i) = \sum_{i=1}^{n} u_i \sum_{j=1}^{m} l_{ij} b_j \ ,$$

where l_{ij} denotes the j-th coefficient of $L(a_i)$ w.r.t. the basis $(b_1, \dots, b_n)$. Hence, L can be represented as $L = (L_{ij})$, $i \in \{1, \dots, m\}$, $j \in \{1, \dots, n\}$, leading to matrix notation

$$v = L(u) = L \cdot u \ .$$

In this form, L and its matrix representation are identified. The dot represents the matrix-vector product

$$(L \cdot u)_i = \sum_{j=1}^{m} l_{ij} u_j \ .$$

The space of all linear maps from U to V is denoted by $\mathrm{Hom}(U, V)$. It can be shown that, given a basis for each U and V, every $L \in \mathrm{Hom}(U, V)$ can be written in the form of an $(m \times n)$-matrix. The algebraic structure of $\mathrm{Hom}(U, V)$ is reflected by the rules of matrix algebra.

We next consider the effect of a basis change on the vector representation of $u \in U$. Let $(\tilde{a}_1, \dots, \tilde{a}_n)$ another basis of U, then

$$a_i = \sum_{j=1}^{n} p_{ij} \tilde{a}_j \qquad \forall i \in \{1, \dots, m\} \ .$$

Hence, the representation $\tilde{u}$ of u w.r.t. $\{\tilde{a}_1, \dots, \tilde{a}_n\}$ is given through

$$u = \sum_{i=1}^{n} u_i a_i =: \sum_{i=1}^{n} u_i \sum_{j=1}^{n} p_{ij} \tilde{a}_j \ ,$$

or equivalently,

$$\tilde{u}_j = \sum_{i=1}^{n} p_{ij} u_i \ .$$

In other words, the (p_{ij}) represent a linear transform $P \in \mathrm{Hom}(U, U)$. P is called a basis transform, and

$$\tilde{u} = P \cdot u \ .$$

Since $\{a_1, \dots, a_n\}$ and $\{\tilde{a}_1, \dots, \tilde{a}_n\}$ can be exchanged in the above calculation, the transformation is invertible, and we easily see that

$$P \cdot P^{-1} = P^{-1} \cdot P = I \ ,$$

with the identity map I. The linear map L can be transformed to another basis in a similar fashion as the elements of U (and V). Let $\{\tilde{b}_1, \ldots, \tilde{b}_m\}$ another basis of V and Q the corresponding basis change matrix, then

$$\tilde{L} \ = \ Q \cdot L \cdot P^{-1}$$

denotes L w.r.t. the bases $\{\tilde{a}_i\}$ and $\{\tilde{b}_j\}$. As a special case, if $U = V$, it is

$$\tilde{L} = P \cdot L \cdot P^{-1} \ .$$

1.1.2 Dual Spaces

Let U an n-dimensional vector space. A linear map

$$\alpha : \ U \to \mathbb{F}$$

is called a *linear form* on U. The set of all linear forms on U is called the *dual space* and noted U^*. As $\mathbb{F}$ is implicitly a vector space, any $\alpha \in U^*$ possesses a representation as a $(1 \times n)$-matrix. For a given basis $\{a_1, \ldots, a_n\}$ of U and the notation from above, linearity mandates

$$\alpha(u) = \sum_{i=1}^{m} \alpha_i \left(a_i u_i\right) \ .$$

It can be shown that U^* is again an n-dimensional vector space, and there exists a *dual basis* $\{a^1, \ldots, a^n\}$ of U^* with the property

$$a^i \left(a_j\right) = \delta_{ij}$$

where δ_{ij} is the Kronecker symbol. If U is equipped with a symmetric (hermitean in the complex case) and positive inner product $< \cdot, \cdot >$, then the *Riesz representation theorem* guarantees to every $\alpha \in U^*$ the existence of $u_\alpha \in U$ such that

$$\alpha(u) =< u_\alpha \, , \, u > \qquad \text{for any } u \in U \ .$$

Similarly, for arbitrary $u \in U$, $\alpha_u :=< u \, , \cdot >$ is an element of U^*. Hence, the two spaces are isomorphic. Furthermore, it is easily seen that $V^{**} = (V^*)^*$ is also isomorphic to V. Commonly, elements of the dual space are written as row vectors.

A linear map $A : U \to V$ between vector spaces gives rise to the *adjoint* map $A^* : V^* \to U^*$ that is defined via

$$(A^* \alpha)(u) := \alpha(Au) \qquad \text{for all } \alpha \in U^*, u \in U \ .$$

A^* is obviously linear, and in the dual bases of U^* and V^* its matrix representation is given as

$$(A^*)_{ij} = (A_{ji})^* \ ,$$

where the star on the right hand side denotes complex conjugation. In other words, the matrix representation of A^* is the conjugate transpose of the matrix of A. In the special case $U = V$, A is called self-adjoint if the matrix representations of A and A^* coincide. In tensor calculus, the concept of dual spaces is found in the occurrence of upper and lower indices, referring to dual or primal properties of a tensor, respectively. These considerations are detailed in Sect. 1.3.

1.1.3 Eigenvalues and Eigenvectors

Let U an n-dimensional vector space and $L \in \mathrm{Hom}(U,U)$ a linear map on U. A vector $u \in U$ is called an *eigenvector* of L to the *eigenvalue* $\lambda \in \mathbb{F}$ if

$$L(u) = \lambda u$$

holds true. Clearly, any scalar multiple of u is also an eigenvector of L to λ, and the eigenvectors to λ form a linear subspace of U, the *eigenspace* E_λ to U. It can be shown that the eigenvalues of any linear map are the roots of its *characteristic polynomial*

$$\chi_L(\lambda) := \det(L - \lambda I) \ ,$$

where $I \in Hom(U,U)$ is again the identity map. The characteristic polynomial is of degree n, which implies that there are at most n eigenvalues of L. If an eigenvalue λ is known, E_λ can be determined as the set of solutions v to the equation

$$(L - \lambda I)(v) = 0 \ .$$

In the case $\mathbb{F} = \mathbb{R}$, the characteristic polynomial has n real roots $\lambda_1, \ldots, \lambda_n$ already if the matrix representation of L in some basis is symmetric, i.e. if $l_{ij} = l_{ji}$ for $i, j \in \{1, \ldots, n\}$. We then say that L is *diagonalizable*, because U can be decomposed into an orthogonal sum of eigenspaces

$$U = E_{\lambda_1} \oplus \cdots \oplus E_{\lambda_n} \ ,$$

giving rise to an orthonormal basis $\{u_1, \ldots u_n\}$ of eigenvectors of U. If P is the corresponding basis transform matrix, then L takes the very simple diagonal matrix representation

$$P \cdot L \cdot P^{-1} = \mathrm{diag}(\lambda_1, \ldots, \lambda_n) \ .$$

In this basis, the properties of the map L are most easily comprehended.

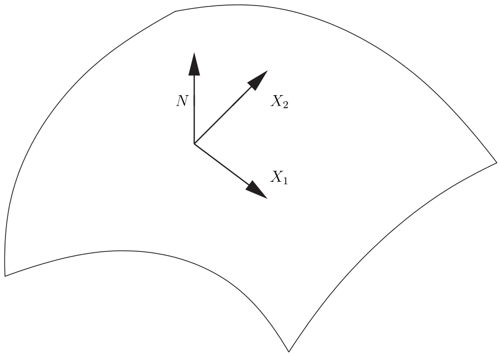

Fig. 1.1. The Gauss frame of a surface

1.2 Fundamentals of Differential Geometry

Definition 1. *A parametrized C^r-surface is a C^r-differential map $X : U \to \mathbb{E}^3$ of an open domain $U \subset \mathbb{R}^2$ into the Euclidean space $\mathbb{E}^3$, whose differential dX is one-to-one for each $u \in U$.*

Remarks. 1. Let $X : U \to \mathbb{E}^3$ be a parametrized surface. A change of variables of X is a diffeomorphism $\tau : \tilde{U} \to U$, where $\tilde{U}$ is an open domain in $\mathbb{R}^2$, such that $d\tau$ has always rank 2. If $\det(\tau^*) > 0$, τ is orientation-preserving (τ^* is the Jacobian matrix of τ). Relationship by change of variables defines an equivalence relation on the class of all parametrized surfaces. A corresponding equivalence class of parametrized surfaces is called a surface in $\mathbb{E}^3$.

2. In this context, the differential dX is a linear map from the tangent space (introduced below) at a point u into $\mathbb{R}$. It is one-to-one if and only if $\partial X/\partial u^1$ and $\partial X/\partial u^2$ are linearly independent at p.

Definition 2. *The two-dimensional subspace T_uX of $\mathbb{E}^3$ generated by* span(X_1, X_2) *is called the* tangent space *of X at u ($X_i := \frac{\partial X}{\partial u^i}$; $i = 1, 2$). Elements of T_uX are called* tangent vectors. *The vector field $N := \frac{[X_1, X_2]}{||X_1|| \cdot ||X_2||}$ is called* unit normal field *($[\cdot,\cdot] : \mathbb{E}^3 \times \mathbb{E}^3 \to \mathbb{E}^3$ is the vector product of $\mathbb{E}^3$). The map N from U to the unit sphere S^2 is called Gauss map and the moving frame is called the Gauss frame of the surface in $\mathbb{E}^3$.*

Remarks. 1. The Gauss frame is in general not an orthogonal frame.

2. Every tangential vector field Y along the surface $X : U \to \mathbb{E}^3$ can be represented in the following form:

$$Y(s) = \lambda^2(s)X_1(u^1(s), u^2(s)) + \lambda^2(s)X_2(u^1(s), u^2(s))$$

Definition 3. *Let $X : U \to \mathbb{E}^3$ be a surface and $u \in U$. The bilinear form I_u on T_uX induced by the inner product of $\mathbb{E}^3$ by restriction is called the* first fundamental form *of the surface.*

Remarks. 1. The matrix representation of the first fundamental form with respect to the basis $\{X_1, X_2\}$ of T_uX is given by

$$\begin{bmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{bmatrix} = \begin{bmatrix} \langle X_1, X_1\rangle & \langle X_1, X_2\rangle \\ \langle X_2, X_1\rangle & \langle X_2, X_2\rangle \end{bmatrix}.$$

Here, $\langle\cdot,\cdot\rangle : \mathbb{E}^3 \times \mathbb{E}^3 \to \mathbb{R}$ is the scalar product.

2. The first fundamental form is symmetric, positive definite and a geometric invariant.
3. Geometrically, the first fundamental form allows us to make measurements on the surface (lengths of curves, angles of tangent vectors, areas of regions) without referring back to the ambient space $\mathbb{E}^3$.

Definition 4. *(a) Let $X : U \to \mathbb{E}^3$ be a surface and $u \in U$. The linear map $L : T_uX \to T_uX$ defined by*

$$L := -dN_u \cdot dX_u^{-1}$$

is called the Weingarten map.

(b) The bilinear form II_u defined by

$$II_u(A, B) := \langle L(A), B\rangle$$

for each $A, B \in T_uX$ is called the second fundamental form *of the surface.*

Remarks. 1. The matrix representation of II_u with respect to the canonical basis of $T_u\mathbb{R}^2$ (identified with $\mathbb{E}^2$) and the associated basis $\{X_1, X_2\}$ of T_uX is given by

$$h_{ij} := \langle -N_i, X_j\rangle = \langle N, X_ij\rangle \qquad i, j \in \{1, 2\}\,.$$

2. The second fundamental form is invariant under congruences of $\mathbb{E}^3$ and orientation-preserving changes of variables.

Proposition and Definition 1. Let $X : U \to \mathbb{E}^3$ be a surface.

(a) The Weingarten map L is self-adjoint. The eigenvalues k_1, k_2 are therefore real and the corresponding eigenvectors are orthogonal.

(b) k_1, k_2 are called the *principal curvatures* of the surface.

(c) The quantity

$$K := k_1 \cdot k_2 = \det(L) = \frac{\det \mathrm{II}}{\det \mathrm{I}}$$

is called the *Gauss curvature* and

$$H := \frac{1}{2}\ \mathrm{trace}(L) = \frac{1}{2}(k_1 + k_2)$$

is called the *mean curvature.*

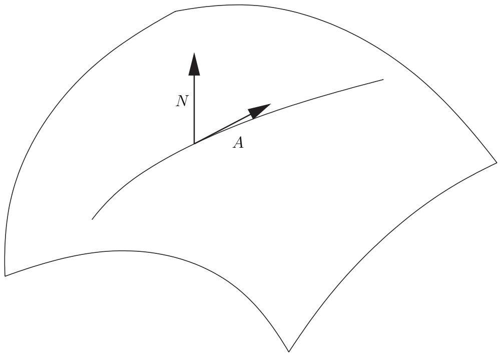

Fig. 1.2. A normal section curve on a surface

Considering surface curves we get to know the geometric interpretations of the second fundamental form: Let $A := \lambda^1 X_1 + \lambda^2 X_2$ be a tangent vector with $||A|| = 1$. Intersecting the surface with the plane given by N and A, we get an intersection curve $y(s)$ with the properties

$$y'(s) = A \qquad \text{and} \qquad e_2 \ = \pm N \ .$$

(e_2 is the principal normal vector of the space curve y). The implicit function theorem implies the existence of this so-called normal section curve. To calculate the extreme values of the curvature of a normal section curve (the so-called normal section curvature) we can use the method of Lagrange multipliers, because we are looking for the extreme values of the normal section curvature k_N under the condition

$$\sum_{i,j=1}^{2} g_{ij}\lambda^i\lambda^j = ||y'(s)|| \ = 1 \ .$$

As the result of these considerations we get:

Proposition 1. Let $X : U \to \mathbb{E}^3$ be a surface and for a tangent vector $A := \lambda^1 X_1 + \lambda^2 X_2$ let $k_N(\lambda^1, \lambda^2)$ be the normal section curvature.

(a)

$$k_N(\lambda^1, \lambda^2) = \sum_{i,j} \frac{h_{ij}\lambda^i\lambda^j}{g_{ij}\lambda^i\lambda^j}$$

(b) Unless the normal section curvature is the same for all directions, there are two perpendicular directions A_1 and A_2 in which k_N attains its absolute maximum and its absolute minimum. These directions and the corresponding normal section curvatures are k_1 and k_2, the *principal curvatures* of the surface.

Proposition and Definition 2. Let $X : U \to \mathbb{E}^3$ be a surface. If $I \subset \mathbb{R}$ is an interval on the real line, let $y : I \to \mathbb{E}^3$ be a surface curve. We denote by $\hat{y}(t)$ the orthogonal projection of $y(t)$ on the tangent plane to X at (an arbitrary) point p.

(a) The *geodesic curvature* k_g of y at p is defined to be the curvature of the projected curve $\hat{y}(t)$ at p.
(b) A curve $y(t)$ on a surface X is called a geodesic curve or simply geodesic if its geodesic curvature k_g vanishes identically.
(c)

$$k_g := \det(y', y'', N) \ ,$$

where the dots denote derivatives with respect to the arc length s of y.

The great importance of tensor fields is totally obvious by just looking briefly on these facts and results from differential geometry. This motivates to have closer look on tensors in general.

For question no touched upon here, the book by do Carmo [2] provides an detailed introductory-level treatment of differential geometry. For a more comprehensive overview, see [3].

1.3 Tensor Fields – A Mathematical Concept

Tensor fields are invariant under parameter transformation and therefore an appropriate tool to describe certain 'geometric invariant situations'. We are using Einstein's summation convention for simplicity (if an index occurs more than once in the same expression, the expression is implicitly summed over all possible values for that index).

Definition 5 ((r,s)-tensor).

(a) V is an n-dimensional vector space and V^* is its dual space. A multilinear map

$$T : \underbrace{V \times \cdots \times V}_{r} \times \underbrace{V^* \times \cdots \times V^*}_{s} \ \to \ \mathbb{R}$$

is called an (r,s)-tensor*.*
(b) $E_1, \ldots, E_n$ is the basis of V, and $E^1, \ldots, E^n$ is the dual basis of V^*. Then

$$\begin{aligned} & T(A_1, \ldots, A_r, B^1, \ldots, B^s) \\ &= T(a_1^{i_1} E_{i_1}, \ldots, a_r^{i_r} E_{i_r}, b_{j_1}^1 E^{j_1}, \ldots, b_{j_s}^s E^{j_s}) \\ &= T(E_{i_1}, \ldots, E_{i_r}, E^{j_1}, \ldots, E^{j_s}) \ a_1^{i_1} \cdots a_r^{i_r} \cdot b_{j_1}^1 \cdots b_{j_s}^s \\ &= t_{i_1 \ldots i_r}{}^{j_1 \ldots j_s} \ \cdot \ a_1^{i_1} \cdots a_r^{i_r} \cdot b_{j_1}^1 \cdots b_{j_s}^s . \end{aligned}$$

The n^{r+s} real numbers are called components of the (r,s)-tensor T.

Under a change of basis

$$\bar{E}_i = \alpha_i{}^j \, E_j \qquad \text{and} \qquad \bar{E}^i = \bar{\alpha}^i{}_j \, E^j$$

an (r, s)-tensor transforms in this way:

$$t_{i_1 \dots i_r}{}^{j_1 \dots j_s} = t_{l_1 \dots l_r}{}^{k_1 \dots k_s} \;\cdot\; \alpha_{i_1}{}^{l_1} \cdots \alpha_{i_r}{}^{l_r} \cdot \bar{\alpha}_{k_1}{}^{j_1} \cdots \bar{\alpha}_{k_s}{}^{j_s} \, .$$

Remarks. 1. The elements of $\mathbb{R}$ are tensors of type $(0, 0)$.
2. The elements of V^* are tensors of type $(1, 0)$.
3. The elements of V can be identified with tensors of type $(0, 1)$, since V^{**} and V are canonically isomorphic.
4. The determinant on V as a multilinear map from V^n to $\mathbb{R}$ is a prototypical example of a $(n, 0)$-tensor.

Tensor operations:

1. Tensors of the same type can be added and scaled like vectors.
2. **Tensor product:**

   Let T an (r, s)-tensor and $\tilde{T}$ a $(\tilde{r}, \tilde{s})$-tensor. Then

$$\begin{aligned} T \circ \tilde{T}(A_1, \dots, A_{r+\tilde{r}}, B^1, \dots, B^{s+\tilde{s}}) \quad &:= \\ T(A_1, \dots, A_r, B^1, \dots, B^s) \;\cdot\; &\tilde{T}(A_{r+1}, \dots, A_{r+\tilde{r}}, B^{s+1}, \dots, B^{s+\tilde{s}}) \end{aligned}$$

   is called the tensor product of T and $\tilde{T}$. In components:

$$(t\tilde{t})_{i_1 \dots i_{r+\tilde{r}}}{}^{j_1 \dots j_{s+\tilde{s}}} \;:=\; t_{i_1 \dots i_r}{}^{j_1 \dots j_s} \cdot t_{i_{r+1} \dots i_{r+\tilde{r}}}{}^{j_{s+1} \dots j_{s+\tilde{s}}} \, .$$

   Clearly, $T \circ \tilde{T}$ is a $(r + \tilde{r}, s + \tilde{s})$-tensor.
3. **Contraction of a tensor:**

$$(r, s)\text{-tensor} \;\left|\; \begin{array}{ccc} T & \to & \tilde{T} \\ t_{i_1 \dots i_r}{}^{j_1 \dots j_s} & \mapsto & t_{i\, i_2 \dots i_r}{}^{i\, j_2 \dots j_s} \end{array} \;\right|\; (r-1, s-1)\text{-tensor}$$

   Example: $\{g_{ij}\}$ and $\{h_{ij}\}$ are the components of the first and second fundamental forms of a surface and H is the mean curvature.

$$g_{ij}, h_{rs} \underset{\text{tensor product}}{\longrightarrow} g^{ij} h_{rs} \underset{\text{contraction}}{\longrightarrow} g^{ij} h_{is} \underset{\text{contraction}}{\longrightarrow} g^{ij} h_{ij} \;=\; 2H$$

4. Inner multiplication with a metric tensor: $\{g_{ij}\}$ are the components of a non-degenerate metric tensor.

$$\begin{aligned} g^{ij} \, t_{j\, i_2 \dots i_r}{}^{j_1 \dots j_s} \;&=:\; t^i{}_{i_2 \dots i_r}{}^{j_1 \dots j_s} \\ g_{ik} \, t_{i_1 \dots i_r}{}^{k\, j_2 \dots j_s} \;&=:\; t_{i_1 \dots i_r\, k}{}^{j_2 \dots j_s} \end{aligned}$$

   Example: the Weingarten map $h_i{}^j = h_{ir} g^{rs}$.

The tensor concept is by no means restricted to differential geometry.

Examples. 1. In the context of physics, the fundamental characteristic that affects the deformation of materials is called *stress*. Since the behavior of a material does not depend on the coordinates used in its description, stress can be described by the *stress tensor* T. The name 'tensor' originates in this context since it was first used to describe tension (stress).
Considering an infinitesimal volume element of a certain material, the stress tensor describes the force that is necessary to establish an equilibrium condition in the material. It depends linearly on the normal of the surface on which the force acts. In turn, force is given as vector, hence the stress tensor can be written in the form of a matrix, which is a tensor of order 2.
In three dimensions of space, given an orthogonal coordinate system e_x, e_y, e_z, the stress tensor components are given with respect to the planes normal to the coordinate axes. For each plane, three scalars describe the required force. Therefore, the tension t with respect to a plane normal to n is given as

$$t = T(n) \;=\; \begin{pmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{pmatrix} \cdot n$$

Here, for example, σ_{xz} denotes the stress on the plane normal to e_x in direction of e_z.
In many applications, T is a symmetric tensor and therefore possesses an orthogonal system of eigenvectors. They are called *principal stress directions.* In this coordinate system, the off-diagonal components of T vanish, and the volume can be held in equilibrium by forces parallel to the principal directions.

2. In medical applications, Diffusion Tensor Imaging is playing an important role. Using Magnetic Resonance Imaging (MRI), it is possible to record the directional diffusion behavior of water in the brain. This allows to deduce information about structures of interest. Again, the diffusion properties are independent of the reference frame used to describe them.
Mathematically speaking, the diffusion tensor T is a second-order 3×3-tensor that maps a direction d to the directional diffusion coefficient c via

$$c(d) = d^T T d \qquad \text{with} \qquad T = \begin{pmatrix} t_{xx} & t_{xy} & t_{xz} \\ t_{yx} & t_{yy} & t_{yz} \\ t_{zx} & t_{zy} & t_{zz} \end{pmatrix} .$$

The diffusion tensor is symmetric and positive semidefinite, which implies that its eigenvalues are real and non-negative. Roughly speaking, the homogeneity of the eigenvalues is a measure of the isotropy of the material with respect to diffusion. If one of the eigenvalues is essentially larger than the others, the corresponding eigenvector indicates the preferred diffusion direction. Looking for locations with anisotropic diffusion tensors, it is possible to identify the direction of certain structures (mostly fibers) in the brain.

We refer the reader to the books by Borisenko and Tarpov [4] and Abraham, Marsden and Ratiu [5] for more detailed introductions and examples to the general topic of tensor calculus and analysis.

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Part I

Feature Detection with Tensors

2

Adaptive Structure Tensors and their Applications

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Summary. The structure tensor, also known as second moment matrix or Förstner interest operator, is a very popular tool in image processing. Its purpose is the estimation of orientation and the local analysis of structure in general. It is based on the integration of data from a local neighborhood. Normally, this neighborhood is defined by a Gaussian window function and the structure tensor is computed by the weighted sum within this window. Some recently proposed methods, however, adapt the computation of the structure tensor to the image data. There are several ways how to do that. This chapter wants to give an overview of the different approaches, whereas the focus lies on the methods based on robust statistics and nonlinear diffusion. Furthermore, the data-adaptive structure tensors are evaluated in some applications. Here the main focus lies on optic flow estimation, but also texture analysis and corner detection are considered.

2.1 Introduction

Orientation estimation and local structure analysis are tasks that can be found in many image processing and early vision applications, e.g. in fingerprint analysis, texture analysis, optic flow estimation, and in geo-physical analysis of soil layers. The classical technique to estimate orientation is to look at the set of luminance gradient vectors in a local neighborhood. This leads to a very popular operator for orientation estimation, the matrix field of the so-called *structure tensor* [4, 10, 16, 20, 38].

The concept of the structure tensor is a consequence of the fact that one can only describe the local structure at a point by considering also the data of its neighborhood. For instance, from the gradient at a single position, it is not possible to distinguish a corner from an edge, while the integration of the gradient information in the neighborhood of the pixel gives evidence about whether the pixel is occupied by an edge or a corner. Further on, the consideration of a local neighborhood becomes even more important as soon as the data is corrupted by noise or other disturbing artifacts, so that the structure has to be estimated before the background of unreliable data.

The structure tensor therefore extends the structure information of each pixel, which is described in a first order approximation by the gradient at that pixel, by the structure information of its surroundings weighted with a Gaussian window function. This comes down to the convolution of the structure data with a Gaussian kernel, i.e. Gaussian smoothing.

Note however, that the smoothing of gradients can lead to cancellation effects. Consider, for example, a thin line. At one side of the line there appears a positive gradient, while at the other side the gradient is negative. Smoothing the gradients will cause them to mutually cancel out. This is the reason why in the structure tensor, the gradient is considered in form of its outer product. The outer product turns the gradient vector ∇I of an image I into a symmetric positive semi-definite matrix, which we will refer to as the *initial matrix field*

$$J_0 := \nabla I \nabla I^\top = \begin{pmatrix} I_x^2 & I_x I_y \\ I_x I_y & I_y^2 \end{pmatrix} . \tag{2.1}$$

Subscripts thereby denote partial derivatives. The structure tensor can be easily generalized from scalar-valued data to vector-valued data. As with the matrix representation it is possible to sum up gradient information, the structure information from all channels of a vector-valued image $\mathbf{I} = (I_1, \ldots, I_N)$ can be integrated by taking the sum of all matrices [8]:

$$J_0 := \sum_{i=1}^{N} \nabla I_i \nabla I_i^\top . \tag{2.2}$$

The structure tensor for a certain neighborhood of scale ρ is then computed by convolution of the components of J_0 with a Gaussian kernel K_ρ:

$$J_\rho = K_\rho * J_0 . \tag{2.3}$$

The smoothing, i.e. the integration of neighborhood information, has two positive effects on orientation estimation. Firstly, it makes the structure tensor robust against noise or other artifacts, and therefore allows a more reliable estimation of orientation in real-world data. Secondly, it distributes the information about the orientation into the areas between edges. This is a very important effect, as it allows to estimate the dominant orientation also at those points in the image where the gradient is close to zero. The dominant orientation can be obtained from the structure tensor as the eigenvector to the largest eigenvalue. An operator which is closely related to the structure tensor is the boundary tensor discussed in Chap. 4 by Köthe.

There are many applications for the structure tensor in the field of image processing. One popular application is optic flow estimation based on the local approach of Lucas and Kanade [21]. In optic flow estimation one searches for the spatio-temporal direction with least change in the image, which is the eigenvector to the *smallest* eigenvalue of the structure tensor [4, 15].

Another application for orientation estimation is texture analysis. Here the dominant orientation extracted from the structure tensor can serve as a feature to discriminate textures [4, 28]. The dominant local orientation is also used in order to drive anisotropic diffusion processes, which enhance the coherence of structures [39]. Often the structure tensor is also used as a feature detector for edges or corners [10]. An application apart from image processing is a structure analysis for grid optimization in the scope of fluid dynamics [34].

Although the classic structure tensor has proven its value in all these applications, it also holds a drawback. This becomes apparent as soon as the orientation in the local neighborhood is not homogeneous like near the boundary of two different textures or two differently moving objects. In these areas, the local neighborhood induced by the Gaussian kernel integrates ambiguous structure information that actually does not belong together and therefore leads to inaccurate estimations.

There are two alternatives to remedy this problem. One is to adapt the neighborhood to the data. A classical way of doing so is the Kuwahara-Nagao operator [2, 18, 25]. At a certain position in an image this operator searches for a nearby neighborhood where the response (the orientation) is more homogeneous than it is at the border. That response is then used at the point of interest. In this way the neighborhoods are not allowed to cross the borders of the differently oriented regions. In [36] it was shown that the classic Kuwahara-Nagao operator can be interpreted as a ‘macroscopic’ version of a PDE image evolution that combines linear diffusion (smoothing) with morphological sharpening (a shock filter in PDE terms). A very similar approach is to use adaptive Gaussian windows [23, 26] for choosing the local neighborhood. Also by nonlinear diffusion one can perform data-adaptive smoothing that avoids the integration of ambiguous data [7, 41].

A second possibility to enhance local orientation estimation is to keep the non-adaptive window, but to clearly choose one of the ambiguous orientations by means of robust statistics [37]. This chapter will describe both approaches and will show their performance in the most common applications also in comparison to the conventional structure tensor. Note that for a data-adaptive structure tensor to reveal any advantages, discontinuities or mixed data must play a role for the application. Some applications where this is the case are optic flow estimation, texture discrimination, and corner detection.