Trends in Mathematics

Fabrizio Colombo
Irene Sabadini
Daniele C. Struppa
Mihaela B. Vajiac
Editors

Advances in Complex Analysis and Operator Theory

Festschrift in Honor of Daniel Alpay's 60th Birthday

Birkhäuser

Birkhäuser

Trends in Mathematics

Trends in Mathematics is a series devoted to the publication of volumes arising from conferences and lecture series focusing on a particular topic from any area of mathematics. Its aim is to make current developments available to the community as rapidly as possible without compromise to quality and to archive these for reference.

Proposals for volumes can be submitted using the Online Book Project Submission Form at our website www.birkhauser-science.com.

Material submitted for publication must be screened and prepared as follows:

All contributions should undergo a reviewing process similar to that carried out by journals and be checked for correct use of language which, as a rule, is English. Articles without proofs, or which do not contain any significantly new results, should be rejected. High quality survey papers, however, are welcome.

We expect the organizers to deliver manuscripts in a form that is essentially ready for direct reproduction. Any version of TEX is acceptable, but the entire collection of files must be in one particular dialect of TEX and unified according to simple instructions available from Birkhäuser.

Furthermore, in order to guarantee the timely appearance of the proceedings it is essential that the final version of the entire material be submitted no later than one year after the conference.

More information about this series at http://www.springer.com/series/4961

Fabrizio Colombo • Irene Sabadini • Daniele C. Struppa
Mihaela B. Vajiac
Editors

Advances in Complex Analysis and Operator Theory

Festschrift in Honor of Daniel Alpay's 60th Birthday

Birkhäuser

Editors
Fabrizio Colombo
Dipartimento di Matematica
Politecnico di Milano
Milano, Italy

Irene Sabadini
Dipartimento di Matematica
Politecnico di Milano
Milano, Italy

Daniele C. Struppa
Schmid College of Science and Technology
Chapman University
Orange, CA, USA

Mihaela B. Vajiac
Schmid College of Science and Technology
Chapman University
Orange, CA, USA

ISSN 2297-0215 ISSN 2297-024X (electronic)
Trends in Mathematics
ISBN 978-3-319-62361-0 ISBN 978-3-319-62362-7 (eBook)
DOI 10.1007/978-3-319-62362-7

Library of Congress Control Number: 2017954335

Mathematics Subject Classification (2010): 30E05, 46E22, 47A, 30H

Printed on acid-free paper

This book is published under the trade name Birkhäuser, www.birkhauser-science.com
The registered company is Springer International Publishing AG
The registered company address is: Gewerbestrasse 11, 6330 Cham, Switzerland

Preface

Daniel Alpay graduated in 1986 from the Weizmann Institute of Science with the Ph.D. thesis: "Reproducing kernel Krein spaces of analytic functions and inverse scattering" under the supervision of Prof. Harry Dym. After graduating, he occupied positions at the University of Tel Aviv, at the University of Groningen, at the Virginia Polytechnic Institute in Blacksburg, at the Weizmann Institute of Science in Rehovot, and at INRIA in Valbonne. In 1991 he moved to Ben Gurion University in Beer-Sheva, Israel, where he joined the Department of Mathematics. There he became a tenured member in 1995, and his mathematical career took off from there; later, in December 2005, Daniel was awarded the Earl Katz family chair in algebraic system theory. In August 2016 Daniel was offered the Foster G. and Mary McGaw Professorship in Mathematical Sciences at Chapman University where he was received with open arms.

Apart from his own extensive research, Daniel is the initiator and Editor-in-Chief of the journal "Complex Analysis and Operator Theory" published by Birkhäuser whose first issue appeared in January 2007. Since 2005 he is also a co-editor of a sub-series entitled "Linear Operators and Linear Systems" of the book series "Operator Theory: Advances and Applications" also published by Birkhäuser, as well as a member of the editorial boards of four other journals. His prolific career has produced 230 papers and 6 books and counting, with over 70 collaborators all over the world.

Daniel's current research interests include: Schur analysis in the setting of slice-hyperholomorphic functions, infinite dimensional analysis in the white noise space setting, free analysis, rational functions and applications to linear system theory and wavelets, Schur analysis in the setting of bicomplex functions, operator theory and Riemann surfaces, interpolation theory, reproducing kernel methods in one and several complex variables. This, by any means, is not an exhaustive list.

This volume collects contributions written by Daniel's friends and collaborators. Several of them have participated in the conference *International Conference on Complex Analysis and Operator Theory* held in honor of Daniel's 60th birthday at Chapman University in November 2016. We are grateful to all the authors and to the referees who helped us to form this volume.

Fabrizio Colombo
Irene Sabadini
Daniele C. Struppa
Mihaela B. Vajiac

Contents

Classes of de Branges Matrices and Corresponding de Branges Spaces

Damir Z. Arov and Harry Dym

To Daniel on the occasion of his sixtieth birthday, with our best wishes

Abstract. Reproducing kernel Hilbert spaces $\mathcal{B}(\mathfrak{E})$ of vector-valued entire functions and reproducing kernel

$$K_\omega^{\mathfrak{E}}(\lambda) = \frac{E_+(\lambda)E_+(\omega)^* - E_-(\lambda)E_+(\omega)^*}{-2\pi i(\lambda - \overline{\omega})}$$

based on an entire matrix-valued function $\mathfrak{E}(\lambda) = \begin{bmatrix} E_-(\lambda) & E_+(\lambda) \end{bmatrix}$ with $p \times p$ blocks $E_\pm(\lambda)$ were introduced and extensively studied by Louis de Branges. In this paper a new subclass of the matrices $\mathfrak{E}(\lambda)$ is introduced and its relation to other subclasses that were presented earlier is discussed.

Mathematics Subject Classification (2000). 46E22, 47B32, 30H99.

Keywords. de Branges spaces, de Branges matrices, reproducing kernels, entire matrix-valued inner functions, reproducing kernel Hilbert spaces.

1. Introduction

An entire $p \times 2p$ mvf (matrix-valued function) $\mathfrak{E}(\lambda) = \begin{bmatrix} E_-(\lambda & E_+(\lambda) \end{bmatrix}$ with $p \times p$ blocks $E_\pm(\lambda)$ will be called an **entire dB (de Branges) matrix** if

(1) $\det E_+(\lambda) \not\equiv 0$ in the complex plane $\mathbb{C}$ and
(2) the mvf $\chi(\lambda) = (E_+^{-1}E_-)(\lambda)$ is holomorphic and contractive in the open upper half plane $\mathbb{C}_+$ and unitary on the line $\mathbb{R}$.

Since $E_\pm(\lambda)$ are entire mvf's, the condition in (2) ensures that

$$E_+(\lambda)E_+^\#(\lambda) - E_-(\lambda)E_-^\#(\lambda) = 0 \quad \text{for every point } \lambda \in \mathbb{C},$$

[1] D.Z. Arov acknowledges with thanks the support of a Morris Belkin Visiting Professorship at the Weizmann Institute.

F. Colombo et al. (eds.), *Advances in Complex Analysis and Operator Theory*,
Trends in Mathematics, DOI 10.1007/978-3-319-62362-7_1

where $f^\#(\lambda) = f(\overline{\lambda})^*$. Moreover, the kernel

$$K_\omega^{\mathfrak{E}}(\lambda) = \frac{E_+(\lambda)E_+(\omega)^* - E_-(\lambda)E_-(\omega)^*}{-2\pi i(\lambda - \overline{\omega})}$$

is positive in the sense of (P3) in Section 3 and hence there exists exactly one RKHS (reproducing kernel Hilbert space) $\mathcal{B}(\mathfrak{E})$ of $p \times 1$ entire vvf's (vector-valued functions) with RK (reproducing kernel) $K_\omega^{\mathfrak{E}}(\lambda)$, i.e., for every point $\omega \in \mathbb{C}$, every u in the space $\mathbb{C}^p$ of complex $p \times 1$ vectors and every $f \in \mathcal{B}(\mathfrak{E})$

(1) $K_\omega^{\mathfrak{E}} u \in \mathcal{B}(\mathfrak{E})$ (as a function of λ) and
(2) $\langle f, K_\omega^{\mathfrak{E}} u\rangle_{\mathcal{B}(\mathfrak{E})} = u^* f(\omega)$.

The space $\mathcal{B}(\mathfrak{E})$ will be called the **dB (de Branges) space** corresponding to the dB matrix $\mathfrak{E}$.

The spaces $\mathcal{B}(\mathfrak{E})$ were first introduced by L. de Branges for the case $p = 1$ in an extensive series of papers on Hilbert spaces of entire functions in the late fifties and early sixties that culminated in the monograph [10]. De Branges also considered generalizations to vector-valued spaces of holomorphic functions in [7], [8], [9] and [11].

The spaces $\mathcal{B}(\mathfrak{E})$ are of fundamental importance in the study of direct and inverse problems for canonical systems of differential and integral equations and numerous other problems of analysis; see, e.g., [2], [3] and [6] and the references cited therein; the review papers [4] and [5] may also be helpful for the reader.

In this paper attention is focused on subclasses of a class $\mathcal{I}(j_p)$ of entire dB matrices $\mathfrak{E}$ for which the corresponding RKHS $\mathcal{B}(\mathfrak{E})$ is invariant under the generalized backwards shift operator. An essential role is played by a pair of entire inner mvf's $\{b_3, b_4\}$ which are characterized by the condition $E_-(\overline{\lambda})^* b_3(\lambda)$ and $b_4(\lambda)E_+(\lambda)$ are outer mvf's. A new subclass $\mathcal{I}_{sR}(j_p)$ of **strongly regular dB matrices** of $\mathcal{I}(j_p)$ is introduced and some of its properties, including its relation to other subclasses of $\mathcal{B}(\mathfrak{E})$ that have been presented earlier are discussed. The main new results are presented in Section 6 and in Theorem 5.3.

The rest of this paper is organized as follows: Supplementary notation is presented in Section 2; RKHS's are reviewed briefly in Section 3;the specific RKHS's $\mathcal{B}(\mathfrak{E})$ based on dB matrices $\mathfrak{E}$ and some of their main properties are discussed in Section 4. Sections 5 and 6 focus on subclasses of $\mathcal{B}(\mathfrak{E})$ and finally Section 7 discusses some connections with a class of entire J_p-inner mvf's $A(\lambda)$ and the corresponding RKHS's $\mathcal{H}(A)$ that were not pursued in this paper and references to other related work that was not treated here.

2. Notation

To proceed further we will need some notation. Here and below $\mathbb{C}$ denotes the complex plane, $\mathbb{C}_+$ (resp., $\mathbb{C}_-$) the open upper (resp., lower) half plane, $\mathbb{R}$ the real axis; mvf (resp., vvf)is an acronym for matrix-valued function (resp., vector-valued function); $M \succeq 0$ (resp., $M \succ 0$) means that M is a positive semi-definite (resp., positive definite) matrix, and cls is an acronym for closed linear span.

The symbols

$$\rho_\omega(\lambda) = -2\pi i(\lambda - \overline{\omega}), \quad f^\#(\lambda) = f(\overline{\lambda})^*,$$

and, for mvf's that are holomorphic in a neighborhood of $\alpha \in \mathbb{C}$,

$$(R_\alpha f)(\lambda) = \begin{cases} \dfrac{f(\lambda) - f(\alpha)}{\lambda - \alpha} & \text{for } \lambda \neq \alpha, \\ f'(\alpha) & \text{for } \lambda = \alpha, \end{cases}$$

will be used, as will the signature matrices

$$j_p = \begin{bmatrix} I_p & 0 \\ 0 & -I_p \end{bmatrix} \quad \text{and} \quad J_p = \begin{bmatrix} 0 & -I_p \\ -I_p & 0 \end{bmatrix}; \tag{2.1}$$

they are unitarily equivalent:

$$J_p = \mathfrak{V} j_p \mathfrak{V}^*, \quad \text{where} \quad \mathfrak{V} = \mathfrak{V}^* = \frac{1}{\sqrt{2}} \begin{bmatrix} -I_p & I_p \\ I_p & I_p \end{bmatrix}. \tag{2.2}$$

A $p \times q$ mvf $f(\lambda)$ belongs to the class:

- $L_2^{p\times q}$ if f is measurable on $\mathbb{R}$ and
$$\|f\|_{\mathrm{st}}^2 = \int_{-\infty}^{\infty} \mathrm{trace}\{f(\mu)^* f(\mu)\} d\mu < \infty;$$
- $H_2^{p\times q}$ (the **Hardy class**) if it is holomorphic in $\mathbb{C}_+$ and if
$$\|f\|_2^2 = \sup_{\nu>0} \int_{-\infty}^{\infty} \mathrm{trace}\{f(\mu + i\nu)^* f(\mu + i\nu)\} d\mu \ < \ \infty;$$
- $(H_2^{p\times q})^\perp$ if $f^\# \in H_2^{q\times p}$ (the superscript $\perp$ is in the notation because $H_2^{p\times q}$ and $(H_2^{p\times q})^\perp$ are orthogonal to each other when regarded as subspaces of $L_2^{p\times q}$);
- $H_\infty^{p\times q}$ if it is holomorphic in $\mathbb{C}_+$ and if
$$\|f\|_\infty = \sup\{\|f(\lambda)\| : \lambda \in \mathbb{C}_+\} < \infty;$$
- $\mathcal{S}^{p\times q}$ (the **Schur class**) if it is in $H_\infty^{p\times q}$ and $\|f\|_\infty \le 1$;
- $\mathcal{S}_{\mathrm{in}}^{p\times q}$ (the class of **inner** $p \times q$ mvf's) if it is in $\mathcal{S}^{p\times q}$ and the limit $f(\mu) = \lim_{\nu\downarrow 0} f(\mu + i\nu)$ (which exists a.e. by a lemma of Fatou) meets the constraint $f(\mu)^* f(\mu) = I_q$ a.e. on $\mathbb{R}$;
- $\mathcal{S}_{\mathrm{out}}^{p\times q}$ (the class of **outer** contractive $p \times q$ mvf's) if it is in $\mathcal{S}^{p\times q}$ and fH_2^q is dense in H_2^p;
- $\mathcal{C}^{p\times p}$ (the **Carathéodory class**) if $q = p$, f is holomorphic in $\mathbb{C}_+$ and
$$(\Re f)(\lambda) \ = \ \frac{f(\lambda) + f(\lambda)^*}{2} \ \succeq \ 0$$
for every point $\lambda \in \mathbb{C}_+$;

- $\mathcal{N}^{p\times q}$ (the **Nevanlinna class** of mvf's with bounded Nevanlinna characteristic) if it can be expressed in the form $f = h^{-1}g$, where $g \in \mathcal{S}^{p\times q}$ and $h \in \mathcal{S} \stackrel{\text{def}}{=} \mathcal{S}^{1\times 1}$;
- $\mathcal{N}_+^{p\times q}$ (the **Smirnov class**) if it can be expressed in the form $f = h^{-1}g$, where $g \in \mathcal{S}^{p\times q}$ and $h \in \mathcal{S}_{\text{out}} \stackrel{\text{def}}{=} \mathcal{S}_{\text{out}}^{1\times 1}$;
- $\mathcal{N}_{\text{out}}^{p\times q}$ (the class of **outer** mvf's in $\mathcal{N}^{p\times q}$) if it can be expressed in the form $f = h^{-1}g$, where $g \in \mathcal{S}_{\text{out}}^{p\times q}$ and $h \in \mathcal{S}_{\text{out}}$;
- $\Pi^{p\times q}$ if it belongs to $\mathcal{N}^{p\times q}$ and there exists a $p\times q$ mvf f_- that is meromorphic in $\mathbb{C}_-$ such that $f_-^{\#} \in \mathcal{N}^{q\times p}$ and $\lim_{\nu\downarrow 0} f(\mu + i\nu) = \lim_{\nu\downarrow 0} f_-(\mu - i\nu)$ a.e. on $\mathbb{R}$.
- $\mathcal{E}^{p\times q}$ if it is an entire $p \times q$ mvf;

  For each class of $p \times q$ mvf's $\mathcal{X}^{p\times q}$ we shall use the symbols

$$\mathcal{X} \quad \text{instead of } \mathcal{X}^{1\times 1} \quad \text{and} \quad \mathcal{X}^p \quad \text{instead of } \mathcal{X}^{p\times 1}; \tag{2.3}$$

$$\mathcal{X}_{\text{const}}^{p\times q} \quad \text{for the set of mvf's in } \mathcal{X}^{p\times q} \text{ that are constant;}$$

$$\mathcal{E} \cap \mathcal{X}^{p\times q} \quad \text{for the class of entire mvf's in } \mathcal{X}^{p\times q}.$$

3. Reproducing kernel Hilbert spaces

A Hilbert space $\mathcal{H}$ of $n \times 1$ vvf's (vector-valued functions) on a set $\Omega \subseteq \mathbb{C}$ is a RKHS if there exists an $n \times n$ mvf $K_\omega(\lambda)$ on $\Omega \times \Omega$ such that for every choice of λ, $\omega \in \Omega$, $\xi \in \mathbb{C}^n$ and $f \in \mathcal{H}$

(1) The vvf $K_\omega\xi \in \mathcal{H}$.
(2) $\langle f, K_\omega\xi\rangle_{\mathcal{H}} = \xi^* f(\omega)$.

The mvf $K_\omega(\lambda)$ is called a RK (reproducing kernel) for $\mathcal{H}$. The following properties of a RKHS are well known and easily checked:

(P1) $K_\alpha(\beta)^* = K_\beta(\alpha)$.
(P2) A RKHS has exactly one RK.
(P3) A RK is positive in the sense that

$$\sum_{i,j=1}^{n} v_j^* K_{\omega_i}(\omega_j) v_i \geq 0$$

for every choice of points $\omega_1, \ldots, \omega_n \in \mathbb{C}$ and vectors $v_1, \ldots, v_n \in \mathbb{C}^p$ and every positive integer n.

(P4) $\|f(\omega)\| \leq \|K_\omega(\omega)\|^{1/2} \|f\|_{\mathcal{H}}$.

Conversely, if $K_\omega(\lambda)$ is a positive kernel on $\Omega \times \Omega$ in the sense of (P3), then, by the matrix version of a theorem of Aronszjan (see, e.g., Theorem 5.2 of [2]),

$$\text{there exists exactly one RKHS} \quad \mathcal{H} \quad \text{with } K_\omega(\lambda) \text{ as its RK.} \tag{3.1}$$

In this paper we shall deal primarily with RKHS's of entire vvf's. The next lemma provides useful necessary and sufficient conditions on a kernel in order for it to be the RK of a RKHS of entire vvf's.

Lemma 3.1. *If $\mathcal{H}$ is a RKHS of $m \times 1$ vvf's on some nonempty open subset Ω of $\mathbb{C}$ with RK $K_\omega(\lambda)$ on $\Omega \times \Omega$, then every vvf $f \in \mathcal{H}$ is holomorphic in Ω if and only if the following two conditions are met:*

(1) *$K_\omega(\lambda)$ is a holomorphic function of λ in Ω for every point $\omega \in \Omega$ and*

(2) *the function $K_\omega(\omega)$ is continuous on Ω.*

Proof. See, e.g., Lemma 5.6 in [2]. □

Example 3.2. The Hardy space H_2^p is a RKHS of $p \times 1$ vvf's that are holomorphic in $\mathbb{C}_+$ with RK

$$K_\omega(\lambda) = \frac{I_p}{\rho_\omega(\lambda)} \quad \text{for } \lambda,\, \omega \in \mathbb{C}_+.$$

Example 3.3. The space $(H_2^p)^\perp$ is a RKHS of $p \times 1$ vvf's that are holomorphic in $\mathbb{C}_-$ with RK

$$K_\omega(\lambda) = -\frac{I_p}{\rho_\omega(\lambda)} \quad \text{for } \lambda,\, \omega \in \mathbb{C}_-.$$

Example 3.4. If $b \in \mathcal{E} \cap \mathcal{S}_{\text{in}}^{p\times p}$, then b is of exponential type, $\det b(\lambda) = e^{i\lambda\tau} b(0)$ for some $\tau \geq 0$ and

$$b^\#(\lambda) b(\lambda) = I_p \quad \text{for every point } \lambda \in \mathbb{C}.$$

Moreover, the space

$$\mathcal{H}(b) = H_2^p \ominus b H_2^p$$

is a RKHS of $p \times 1$ entire vvf's with RK

$$k_\omega^b(\lambda) = \frac{I_p - b(\lambda) b(\omega)^*}{\rho_\omega(\lambda)} \quad \text{for } \lambda \neq \overline{\omega}.$$

Example 3.5. If $b \in \mathcal{E} \cap \mathcal{S}_{\text{in}}^{p\times p}$, then, in view of the preceding example, the kernel

$$\ell_\omega^b(\lambda) = b^\#(\lambda) k_\omega^b(\lambda) b^\#(\omega)^* = \frac{b^\#(\lambda) b^\#(\omega)^* - I_p}{\rho_\omega(\lambda)} \quad \text{for } \lambda \neq \overline{\omega}$$

is also positive in the sense of (P3) and may be identified as the RK for the space

$$\mathcal{H}_*(b) = (H_2^p)^\perp \ominus b^\# (H_2^p)^\perp = b^\# \mathcal{H}(b)$$

of $p \times 1$ entire vvf's.

Example 3.6. If b_3, $b_4 \in \mathcal{E} \cap \mathcal{S}_{\text{in}}^{p\times p}$, then $\mathcal{H}_*(b_4) \oplus \mathcal{H}(b_3)$ is a RKHS of $p \times 1$ entire vvf's with RK

$$\ell_\omega^{b_4}(\lambda) + k_\omega^{b_3}(\lambda) = \frac{b_4^\#(\lambda) b_4^\#(\omega)^* - b_3(\lambda) b_3(\omega)^*}{\rho_\omega(\lambda)} \quad \text{for } \lambda \neq \overline{\omega}.$$

4. Entire de Branges matrices $\mathfrak{E}$ and de Branges spaces $\mathcal{B}(\mathfrak{E})$

Recall that an entire $p \times 2p$ mvf $\mathcal{E}(\lambda) = \begin{bmatrix} E_-(\lambda) & E_+(\lambda) \end{bmatrix}$ with $p \times p$ blocks $E_\pm$ will be called an entire dB matrix if

$$\det E_+(\lambda) \not\equiv 0 \quad \text{and} \quad E_+^{-1} E_- \in \mathcal{S}_{\text{in}}^{p\times p}. \tag{4.1}$$

If $\mathfrak{E}$ is a dB matrix, then the set of entire $p \times 1$ vvf's that meet the constraints

$$E_+^{-1} f \in H_2^p \quad \text{and} \quad E_-^{-1} f \in (H_2^p)^\perp \tag{4.2}$$

is a RKHS $\mathcal{B}(\mathfrak{E})$ with RK

$$K_\omega^{\mathcal{E}}(\lambda) = \begin{cases} \dfrac{E_+(\lambda)E_+(\omega)^* - E_-(\lambda)E_-(\omega)^*}{\rho_\omega(\lambda)} = -\dfrac{\mathfrak{E}(\lambda) j_p \mathfrak{E}(\omega)^*}{\rho_\omega(\lambda)} & \text{if } \lambda \neq \overline{\omega}, \\ -\frac{1}{2\pi i}\{E_+'(\overline{\omega})E_+(\omega)^* - E_-'(\overline{\omega})E_-(\omega)^*\} & \text{if } \lambda = \overline{\omega}, \end{cases} \tag{4.3}$$

with respect to the inner product

$$\langle f, g \rangle_{\mathcal{B}(\mathfrak{E})} = \int_{-\infty}^{\infty} g(\mu)^* \Delta_{\mathfrak{E}}(\mu) f(\mu) d\mu, \tag{4.4}$$

where

$$\Delta_{\mathfrak{E}}(\mu) = \{E_+(\mu)E_+(\mu)^*\}^{-1} = \{E_-(\mu)E_-(\mu)^*\}^{-1}$$

at points $\mu \in \mathbb{R}$ at which $\det E_\pm(\mu) \neq 0$; see, e.g., Section 4.10 in [3] and Section 3.2 in [6].

Remark 4.1. Since $\mathfrak{E}(\lambda)$ is an entire mvf, the second condition in (4.1) implies that

$$E_+(\lambda)E_+^\#(\lambda) - E_-(\lambda)E_-^\#(\lambda) = 0 \quad \text{for every point } \lambda \in \mathbb{C}. \tag{4.5}$$

Moreover, the mvf $\chi = E_+^{-1}E_-$, which belongs to $\mathcal{S}_{\text{in}}^{p\times p}$ by definition, extends as a meromorphic mvf in $\mathbb{C}$ and the formulas

$$V_1 f = E_+^{-1} f \quad \text{and} \quad V_2 f = E_-^{-1} f$$

define unitary operators from $\mathcal{B}(\mathfrak{E})$ onto the RKHS's $\mathcal{H}(\chi)$ and $\mathcal{H}_*(\chi)$, respectively. The RK's of these spaces are related by the formulas

$$K_\omega^{\mathfrak{E}}(\lambda) = E_+(\lambda) k_\omega^\chi(\lambda) E_+(\omega)^* \quad \text{for } \lambda, \omega \in \mathfrak{h}_\chi \tag{4.6}$$

and

$$K_\omega^{\mathfrak{E}}(\lambda) = E_-(\lambda) \ell_\omega^\chi(\lambda) E_-(\omega)^* \quad \text{for } \lambda, \omega \in \mathfrak{h}_{\chi^\#}, \tag{4.7}$$

where $\mathfrak{h}_\chi$ (resp., $\mathfrak{h}_{\chi^\#}$) denotes the domain of holomorphy of χ (resp., $\chi^\#$) in $\mathbb{C}$. $\diamond$

An entire dB matrix $\mathfrak{E}$ belongs to the class $\mathcal{I}(j_p)$ if

$$(\rho_i E_-^\#)^{-1} \in H_2^{p\times p} \quad \text{and} \quad (\rho_i E_+)^{-1} \in H_2^{p\times p}. \tag{4.8}$$

Lemma 4.2. *If $\mathfrak{E} \in \mathcal{I}(j_p)$, then*

(1) $\det E_+(\lambda) \neq 0$ *for every point* $\lambda \in \overline{\mathbb{C}_+}$ *and* E_+^{-1} *is holomorphic in* $\overline{\mathbb{C}_+}$*;* $\det E_-(\lambda) \neq 0$ *for every point* $\lambda \in \overline{\mathbb{C}_-}$ *and* E_-^{-1} *is holomorphic in* $\overline{\mathbb{C}_-}$. .

(2) *The dB space* $\mathcal{B}(\mathfrak{E})$ *is* R_α*-invariant for every point* $\alpha \in \mathbb{C}$.

(3) $\mathfrak{E} \in \mathcal{E} \cap \Pi^{p\times 2p}$ *and* $\mathcal{B}(\mathfrak{E}) \subset \mathcal{E} \cap \Pi^p$.

(4) $R_\alpha E_+\eta \in \mathcal{B}(\mathfrak{E})$ *and* $R_\alpha E_-\eta \in \mathcal{B}(\mathfrak{E})$ *for every* $\eta \in \mathbb{C}^p$ *and* $\alpha \in \mathbb{R}$.

(5) *The subspaces*

$$\mathcal{N}_\alpha \stackrel{\text{def}}{=} \{u \in \mathbb{C}^p : K_\alpha^{\mathfrak{E}}(\alpha)u = 0\} \quad \textit{and} \quad \mathcal{R}_\alpha \stackrel{\text{def}}{=} \{K_\alpha^{\mathfrak{E}}(\alpha)u : u \in \mathbb{C}^p\} \tag{4.9}$$

are independent of α,

$$\mathbb{C}^p = \mathcal{N}_\alpha \oplus \mathcal{R}_\alpha \tag{4.10}$$

and

$$\mathcal{R}_\alpha = \{f(\alpha) : f \in \mathcal{B}(\mathfrak{E})\}. \tag{4.11}$$

(6) $K_\alpha^{\mathfrak{E}}(\alpha) \succ 0 \iff \mathcal{R}_\alpha = \mathbb{C}^p \iff \mathcal{N}_\alpha = \{0\}$.

(7) $K_\omega^{\mathfrak{E}}(\omega) \succ 0$ *for at least one point* $\omega \in \mathbb{C}$ *if and only if* $K_\omega^{\mathfrak{E}}(\omega) \succ 0$ *for every point* $\omega \in \mathbb{C}$.

Moreover, if $\mathfrak{E}$ *is an entire dB matrix and there exists at least one point* $\alpha \in \mathbb{C}$ *such that*

(a) $\mathcal{B}(\mathfrak{E})$ *is* R_α *invariant and* $K_\alpha^{\mathfrak{E}}(\alpha) \succ 0$,

or, there exists a pair of points $\alpha, \beta \in \mathbb{C}^p$ *such that*

(b) $E_+(\alpha)$ *is invertible,* $E_-(\beta)$ *is invertible,* $R_\alpha E_+\eta \in \mathcal{B}(\mathfrak{E})$ *and* $R_\beta E_-\eta \in \mathcal{B}(\mathfrak{E})$ *for every* $\eta \in \mathbb{C}^p$,

then $\mathfrak{E} \in \mathcal{I}(j_p)$.

Proof. See Lemma 3.19 in [6]. □

A nondecreasing $p \times p$ mvf $\sigma(\mu)$ on $\mathbb{R}$ is called a **spectral function** for a dB space $\mathcal{B}(\mathfrak{E})$ based on a dB matrix $\mathfrak{E} \in \mathcal{I}(j_p)$ if

$$\langle g, g\rangle_{\mathcal{B}(\mathfrak{E})} = \int_{-\infty}^{\infty} g(\mu)^*\, d\sigma(\mu)\, g(\mu) \quad \text{for every } g \in \mathcal{B}(\mathfrak{E}).$$

The set of spectral functions for $\mathcal{B}(\mathfrak{E})$ will be denoted $(\mathcal{B}(\mathfrak{E}))_{sf}$. If $\sigma \in (\mathcal{B}(\mathfrak{E}))_{sf}$ is locally absolutely continuous and $\Delta(\mu) = \sigma'(\mu)$ a.e. is such that

$$\langle g, g\rangle_{\mathcal{B}(\mathfrak{E})} = \int_{-\infty}^{\infty} g(\mu)^*\, \Delta(\mu)\, g(\mu) d\mu \quad \text{for every } g \in \mathcal{B}(\mathfrak{E}),$$

then Δ will be called a **spectral density** for $\mathcal{B}(\mathfrak{E})$. Since

$$\mathfrak{E} \in \mathcal{I}(j_p) \implies \int_{-\infty}^{\infty} \frac{\Delta_{\mathfrak{E}}(\mu)}{1+\mu^2} d\mu \quad \text{is finite,}$$

the function

$$\sigma_{\mathfrak{E}}(\mu) = \int_0^{\mu} \Delta_{\mathfrak{E}}(\nu) d\nu$$

is a spectral function for $\mathcal{B}(\mathfrak{E})$ and $\Delta_{\mathfrak{E}}$ is a spectral density for $\mathcal{B}(\mathfrak{E})$.

Lemma 4.3. *If* $\mathfrak{E} \in \mathcal{I}(j_p)$ *and* $K_\omega^{\mathfrak{E}}(\omega) \succ 0$ *for at least one point* $\omega \in \mathbb{C}$, *then*

$$\int_{-\infty}^{\infty} \text{trace}\left\{\frac{d\sigma(\mu)}{1+\mu^2}\right\} < \infty$$

for every $\sigma \in (\mathcal{B}(\mathfrak{E}))_{sf}$.

Proof. Let $f \in \mathcal{B}(\mathfrak{E})$. Then, as

$$\frac{f(i)}{\lambda - i} = \frac{f(\lambda)}{\lambda - i} - (R_i f)(\lambda),$$

it is readily checked that

$$\begin{aligned}
&\int_{-\infty}^{\infty} f(i)^* \frac{d\sigma(\mu)}{1+\mu^2} f(i) \\
&\leq 2\int_{-\infty}^{\infty} f(\mu)^* \frac{d\sigma(\mu)}{1+\mu^2} f(\mu) + 2\int_{-\infty}^{\infty} (R_i f)(\mu)^* d\sigma(\mu)(R_i f)(\mu) \\
&\leq 2\|f\|^2_{\mathcal{B}(\mathfrak{E})} + 2\|R_i f\|^2_{\mathcal{B}(\mathfrak{E})}.
\end{aligned}$$

The asserted bound follows from the fact that $\{f(i) : f \in \mathcal{B}(\mathfrak{E})\} = \mathbb{C}^p$, by Lemma 4.2. □

Example 4.4. If $b_1, b_2 \in \mathcal{E} \cap \mathcal{S}_{\text{in}}^{p\times p}$ and $\varphi \in \mathcal{E}^{p\times p}$ with $\det \varphi(\lambda) \not\equiv 0$, then

$$\mathfrak{E} = \begin{bmatrix} \varphi b_1 & \varphi b_2^{-1} \end{bmatrix} \quad \text{is a dB matrix and} \quad \Delta_{\mathfrak{E}}(\mu) = \{\varphi(\mu)\varphi(\mu)^*\}^{-1} \text{ a.e. on } \mathbb{R}.$$

If, for example, $\varphi(\lambda) = \lambda I_p$, then $\Delta_{\mathfrak{E}}(\mu) = \mu^{-2} I_p$ for $\mu \neq 0$. Moreover,

(1) if $\mathfrak{E} = \begin{bmatrix} b_1 & b_2^{-1} \end{bmatrix}$, then $\mathcal{B}(\mathfrak{E}) = \mathcal{H}_*(b_1) \oplus \mathcal{H}(b_2)$.
(2) if $\mathfrak{E} = \begin{bmatrix} b_1 & I_p \end{bmatrix}$, then $\mathcal{B}(\mathfrak{E}) = \mathcal{H}_*(b_1)$.
(3) if $\mathfrak{E} = \begin{bmatrix} I_p & b_2^{-1} \end{bmatrix}$, then $\mathcal{B}(\mathfrak{E}) = \mathcal{H}(b_2)$. ◇

A theorem of M.G. Krein (see e.g., Theorem 3.91 in [2]) guarantees that if $f \in \mathcal{E} \cap \Pi^{p\times q}$, then f has finite exponential type

$$\tau_f = \limsup_{r\uparrow\infty} \left\{ \frac{\ln \|f(\lambda)\|}{r} : |\lambda| < r \right\} < \infty$$

and that

$$\tau_f = \max\{\tau_f^-, \tau_f^+\} \quad \text{where} \quad \tau_f^{\pm} = \limsup_{\nu\uparrow\infty} \left\{ \frac{\ln \|f(\pm i\nu)\|}{\nu} \right\}.$$

The next example demonstrates that the inclusions

$$\mathcal{I}(j_p) \subset \{\text{dB matrices } \mathfrak{E} \in \mathcal{E} \cap \Pi^{p\times 2p}\} \subset \{\text{dB matrices } \mathfrak{E} \in \mathcal{E}^{p\times 2p}\}$$

are proper.

Example 4.5. Let $\mathfrak{E} = \begin{bmatrix} \varphi b_1 & \varphi b_2^{-1} \end{bmatrix}$ be the dB matrix with $b_1, b_2 \in \mathcal{E} \cap \mathcal{S}_{\text{in}}^{p\times p}$, $\varphi \in \mathcal{E}^{p\times p}$ and $\det \varphi(\lambda) \not\equiv 0$.

$$\text{If } \varphi(\lambda) = \lambda I_p, \text{ then } \mathfrak{E} \in \mathcal{E} \cap \Pi^{p\times 2p} \text{ but } \mathfrak{E} \notin \mathcal{I}(j_p),$$

since $b_2(\varphi\rho_i)^{-1} \notin H_2^{p\times p}$. On the other hand, if

$$\varphi(\lambda) = e^{\lambda^2} I_p \quad \text{then } \mathfrak{E} \text{ is an entire dB matrix but } \mathfrak{E} \notin \Pi^{p\times 2p}$$

because $\mathfrak{E}$ is not of exponential type. ◇

If $\mathfrak{E} \in \mathcal{I}(j_p)$, then, in view of (4.8), there exist a pair of mvf's b_3, $b_4 \in \mathcal{S}_{\rm in}^{p\times p}$ and $\varphi_3, \varphi_4 \in \mathcal{N}_{\rm out}^{p\times p}$ such that

$$(E_-^{\#})^{-1} = b_3\varphi_3 \quad \text{and} \quad (E_+)^{-1} = \varphi_4 b_4. \tag{4.12}$$

The pair $\{b_3, b_4\}$ is uniquely determined by (4.12) up to $p \times p$ constant unitary multipliers on the right and left, respectively. The set

$$\{(b_3u, vb_4) : u \text{ and } v \text{ are unitary } p\times p \text{ matrices}\}$$

is called the set of **associated pairs of** $\mathfrak{E}$ and is denoted $ap(\mathfrak{E})$;

$$\text{the associated pairs of } \mathfrak{E} \in \mathcal{I}(j_p) \text{ are entire mvf's.} \tag{4.13}$$

The verification of (4.13) follows from [1] and the fact that $ap(\mathfrak{E})$ coincides with the set of associated pairs of the second kind of a mvf A that belongs to the class $\mathcal{E} \cap \mathcal{U}(J_p)$ that is considered briefly in Section 7; see especially (7.2) and (7.1) and, for additional discussion, Theorem 4.54 in [2]).

Lemma 4.6. *If* $\mathfrak{E} = \begin{bmatrix} E_- & E_+ \end{bmatrix}$ *belongs to the class* $\mathcal{I}(j_p)$, *then*

(1) $\tau_{\mathfrak{E}}^+ = \max\{\tau_f^+ : f \in \mathcal{B}(\mathfrak{E})\}$ *and* $\tau_{\mathfrak{E}}^- = \max\{\tau_f^- : f \in \mathcal{B}(\mathfrak{E})\}$;

(2) $\tau_{\mathfrak{E}}^+ = \tau_{E_+}^+ = \tau_{b_4}$ *and* $\tau_{\mathfrak{E}}^- = \tau_{E_-}^- = \tau_{b_3}$.

Proof. See Lemma 3.41 in[1] [6]. ☐

Let $\mathcal{U}_{\rm const}(j_p)$ denote the set of $V \in \mathbb{C}^{2p\times 2p}$ such that $V^* j_p V = j_p$.

Lemma 4.7. *If* $\mathfrak{E}$ *and* $\widetilde{\mathfrak{E}}$ *are entire dB matrices and* $K_0^{\mathfrak{E}}(0) \succ 0$, *then*

$$\mathcal{B}(\mathfrak{E}) = \mathcal{B}(\widetilde{\mathfrak{E}}) \iff \widetilde{\mathfrak{E}}(\lambda) = \mathfrak{E}(\lambda)V \quad \textit{for some } V \in \mathcal{U}_{\rm const}(j_p). \tag{4.14}$$

Proof. The implication $\Longleftarrow$ is obvious; a proof of the opposite implication $\Longrightarrow$ is furnished in Theorem 3.22 of [6]. ☐

The set of matrices

$$V_\alpha = \begin{bmatrix} I_p + i\alpha & i\alpha \\ -i\alpha & I_p - i\alpha \end{bmatrix} \quad \text{with } \alpha = \alpha^* \in \mathbb{C}^{p\times p} \tag{4.15}$$

is a subgroup of the multiplicative group $\mathcal{U}_{\rm const}(j_p)$ with the property

$$V_\alpha V_\beta = V_{\alpha+\beta}. \tag{4.16}$$

Lemma 4.8. *If* $V \in \mathcal{U}_{\rm const}(j_p)$, *then*

$$\begin{bmatrix} I_p & I_p \end{bmatrix} V = \begin{bmatrix} I_p & I_p \end{bmatrix} \Longleftrightarrow V = V_\alpha$$

for some Hermitian matrix $\alpha \in \mathbb{C}^{p\times p}$.

Proof. See Lemma 3.11 in [6]. ☐

Let

$$\mathcal{I}^\circ(j_p) = \{\mathfrak{E} \in \mathcal{I}(j_p) : \mathfrak{E}(0) = \begin{bmatrix} I_p & I_p \end{bmatrix}\}.$$

[1] The enumeration of some of the items referred to in [6] might shift a little in the final edition.

Lemma 4.9. *If $\mathfrak{E} = \begin{bmatrix} E_- & E_+ \end{bmatrix}$ is an entire dB matrix, then there exists a matrix $V \in \mathcal{U}_{\text{const}}(j_p)$ such that $(\mathfrak{E}V)(0) = \begin{bmatrix} I_p & I_p \end{bmatrix}$ if and only if $E_+(0)$ is invertible.*

Proof. See Theorem 3.12 in [6]. □

Lemma 4.10. *If $\mathcal{E}$, $\mathfrak{E}_1 \in \mathcal{I}(j_p)$ and $\mathcal{B}(\mathfrak{E}) = \mathcal{B}(\mathfrak{E}_1)$, then $ap(\mathfrak{E}) = ap(\mathfrak{E}_1)$.*

Proof. This is Corollary 3.17 in [6]. □

5. Subclasses of $\mathcal{I}_R(j_p)$

A Hilbert space $\mathcal{H}_1$ is said to be **contractively included** in a Hilbert space $\mathcal{H}_2$ if

$$f \in \mathcal{H}_1 \implies f \in \mathcal{H}_2 \quad \text{and} \quad \|f\|_{\mathcal{H}_2} \leq \|f\|_{\mathcal{H}_1};$$

the indicated inclusion is said to be **isometric** if

$$\|f\|_{\mathcal{H}_2} = \|f\|_{\mathcal{H}_1} \quad \text{for every } f \in \mathcal{H}_1.$$

The notation $\mathcal{H}_1 \sim \mathcal{H}_2$ means that the two Hilbert spaces coincide as vector spaces and have equivalent norms, i.e., there exist a pair of constants $\gamma_1 > 0$ and $\gamma_2 \geq \gamma_1$ such that

$$\gamma_1 \|f\|_{\mathcal{H}_1} \leq \|f\|_{\mathcal{H}_2} \leq \gamma_2 \|f\|_{\mathcal{H}_1}.$$

If $\mathfrak{E} \in \mathcal{I}(j_p)$ and $\{b_3, b_4\} \in ap(\mathfrak{E})$, then $\mathfrak{E}$ belongs to the subclass

$\mathcal{I}_S(j_p)$ if b_3 and b_4 are constant unitary matrices;

$\mathcal{I}_R(j_p)$ if $\mathfrak{E}_1 \in \mathcal{I}(j_p)$, $\{b_3, b_4\} \in ap(\mathfrak{E}_1)$ and $\mathcal{B}(\mathfrak{E}_1) \subseteq \mathcal{B}(\mathfrak{E})$ is included isometrically in $\mathcal{B}(\mathfrak{E})$, then $\mathcal{B}(\mathfrak{E}_1) = \mathcal{B}(\mathfrak{E})$;

$\mathcal{I}_{AR}(j_p)$ if every mvf $\mathfrak{E}_1$ for which $\mathcal{B}(\mathfrak{E}_1) \subseteq \mathcal{B}(\mathfrak{E})$ is isometric belongs to the class $\mathcal{I}_R(j_p)$.

There exist other characterizations of these subclasses. Thus, for example,

$$\mathcal{I}_S(j_p) = \{\mathfrak{E} \in \mathcal{I}(j_p) : \tau_{\mathfrak{E}} = 0\} = \{\begin{bmatrix} E_- & E_+ \end{bmatrix} \in \mathcal{I}(j_p) : E_-^{\#}, E_+ \in \mathcal{N}_{\text{out}}^{p\times p}\}.$$

Example 5.1. If $p(\lambda)$ is a polynomial of degree ≥ 1 with all its zeros in $\mathbb{C}_-$, then the mvf $\mathfrak{E}(\lambda) = \begin{bmatrix} p^{\#}(\lambda)I_p & p(\lambda)I_p \end{bmatrix}$ belongs to the class $\mathcal{E} \cap \mathcal{I}_S(j_p)$.

If b_3, $b_4 \in \mathcal{E} \cap \mathcal{S}_{\text{in}}^{p\times p}$ and $b_4(\lambda)b_3(\lambda)$ is not constant and $p(\lambda)$ is as above, then the mvf $\mathfrak{E}(\lambda) = \begin{bmatrix} p^{\#}(\lambda)b_3(\lambda) & p(\lambda)b_4(\lambda)^{-1} \end{bmatrix}$ belongs to the class $\mathcal{E} \cap \mathcal{I}(j_p)$ but does not belong to the class $\mathcal{I}_S(j_p)$ or to the class $\mathcal{I}_R(j_p)$. ◇

The next theorem will be used to help establish Theorem 5.3. It follows from the connections between dB matrices $\mathfrak{E}$ and J_p-inner mvf's $A(\lambda)$ that will be discussed briefly in Section 7; complete proofs will be presented in [6].

Theorem 5.2. *If $\mathfrak{E} = \mathcal{I}(j_p)$ and $\{b_3, b_4\} \in ap(\mathfrak{E})$, then:*

(1) *Every closed R_0-invariant subspace $\mathcal{L}$ of $\mathcal{B}(\mathfrak{E})$ is equal to a dB space $\mathcal{B}(\mathfrak{E}_1)$ for some dB matrix $\mathfrak{E}_1 \in \mathcal{I}(j_p)$ (and hence $\|f\|_{\mathcal{B}(\mathfrak{E}_1)} = \|f\|_{\mathcal{B}(\mathfrak{E})}$ for every $f \in \mathcal{B}(\mathfrak{E}_1)$).*

(2) *If* $\mathfrak{E}_1 \in \mathcal{I}(j_p)$, $\{b_3^{(1)}, b_4^{(1)}\} \in ap(\mathfrak{E}_1)$ *and* $\mathcal{B}(\mathfrak{E}_1) \subseteq \mathcal{B}(\mathfrak{E})$, *then*

$$(b_3^{(1)})^{-1} b_3 \in \mathcal{E} \cap \mathcal{S}_{\text{in}}^{p\times p} \quad \textit{and} \quad b_4 (b_4^{(1)})^{-1} \in \mathcal{E} \cap \mathcal{S}_{\text{in}}^{p\times p}.$$

(3) *If the mvf's* $b_3^{(1)}$, $b_4^{(1)}$, $(b_3^{(1)})^{-1} b_3$ *and* $b_4 (b_4^{(1)})^{-1}$ *all belong to the class* $\mathcal{E} \cap \mathcal{S}_{\text{in}}^{p\times p}$, *then there exists a dB matrix* $\mathfrak{E}_1 \in \mathcal{I}_R(j_p)$ *such that* $\mathcal{B}(\mathfrak{E}_1) \subseteq \mathcal{B}(\mathfrak{E})$ *isometrically and* $\{b_3^{(1)}, b_4^{(1)}\} \in ap(\mathfrak{E}_1)$. *Moreover, this dB matrix is unique up to a constant* j_p*-unitary factor* V *on the right.*

Theorem 5.3. *If* $\mathfrak{E} \in \mathcal{I}(j_p)$, $\{b_3, b_4\} \in ap(\mathfrak{E})$ *and the four mvf's* $b_3^{(1)}$, $(b_3^{(1)})^{-1} b_3$, $b_4^{(1)}$ *and* $b_4 (b_4^{(1)})^{-1}$ *all belong to the class* $\mathcal{E} \cap \mathcal{S}_{\text{in}}^{p\times p}$, *then:*

(1) $k_\omega^{b_3^{(1)}} \xi \in \mathcal{B}(\mathfrak{E})$ *and* $\ell_\omega^{b_4^{(1)}} \xi \in \mathcal{B}(\mathfrak{E})$ *for every* $\xi \in \mathbb{C}^p$ *and* $\omega \in \mathbb{C}$.

(2) *The space*

$$\mathcal{L} = \text{cls}\{k_\alpha^{b_3^{(1)}} \xi + \ell_\beta^{b_4^{(1)}} \eta : \, \alpha, \beta \in \mathbb{C} \textit{ and } \xi, \eta \in \mathbb{C}^p\} \quad \textit{in } \mathcal{B}(\mathfrak{E}) \tag{5.1}$$

is a dB space $\mathcal{B}(\mathfrak{E}_1)$ *based on a dB matrix* $\mathfrak{E}_1 \in \mathcal{I}(j_p)$ *and hence* $\|f\|_{\mathcal{B}(\mathfrak{E}_1)} = \|f\|_{\mathcal{B}(\mathfrak{E})}$ *for every* $f \in \mathcal{B}(\mathfrak{E}_1)$, *i.e.,* $\mathcal{B}(\mathfrak{E}_1) \subseteq \mathcal{B}(\mathfrak{E})$ *and the inclusion is isometric.*

(3) $\mathfrak{E}_1 \in \mathcal{I}_R(j_p)$ *and* $\{b_3^{(1)}, b_4^{(1)}\} \in ap(\mathfrak{E}_1)$.

(4) $\mathfrak{E} \in \mathcal{I}_R(j_p)$ *if and only if*

$$\mathcal{B}(\mathfrak{E}) = \text{cls}\{k_\alpha^{b_3} \xi + \ell_\beta^{b_4} \eta : \, \alpha, \beta \in \mathbb{C} \textit{ and } \xi, \eta \in \mathbb{C}^p\} \quad \textit{in } \mathcal{B}(\mathfrak{E}). \tag{5.2}$$

Proof. The proof is divided into steps.

1. *If* $b \in \mathcal{E} \cap \mathcal{S}_{\text{in}}^{p\times p}$, *then* $\rho_i k_\omega^b \in H_\infty^{p\times p}$ *and* $\rho_i (\ell_\omega^b)^\# \in H_\infty^{p\times p}$ *for each choice of* $\omega \in \mathbb{C}$.

Since $\rho_i(\lambda) k_\omega^b(\lambda)$ is an entire mvf of λ for each choice of $\omega \in \mathbb{C}$,

$$\|\rho_i k_\omega^b\| \leq c_1 < \infty \quad \text{for } |\lambda - \overline{\omega}| \leq 1.$$

On the other hand, if $\lambda = \overline{\omega} + \beta$ with $|\beta| > 1$, then

$$\left| \frac{\rho_i(\lambda)}{\rho_\omega(\lambda)} \right| = \left| \frac{\lambda + i}{\lambda - \overline{\omega}} \right| = \frac{|\overline{\omega} + \beta + i|}{|\beta|} \leq 1 + |\overline{\omega} + i|.$$

Therefore,

$$\rho_i k_\omega^b = \frac{\rho_i}{\rho_\omega} \{I_p - b b(\omega)^*\} \in H_\infty^{p\times p}.$$

The verification of the second assertion is similar.

2. $(b_3^{(1)})^\# (\rho_i E_-^\#)^{-1} \in H_2^{p\times p}$ *and* $(\rho_i E_+)^{-1} (b_4^{(1)})^\# \in H_2^{p\times p}$. In view of (4.12),

$$E_-^\# = \varphi_3^{-1} b_3^\# \quad \text{and} \quad E_+ = b_4^\# \varphi_4^{-1}$$

with φ_3 and φ_4 in $\mathcal{N}_{\text{out}}^{p\times p}$. Therefore, since

$$(\rho_i E_+)^{-1} \in H_2^{p\times p} \quad \text{and} \quad (\rho_i E_-^\#)^{-1} \in H_2^{p\times p},$$

the mvf's

$$(\rho_i E_+)^{-1} b_4^\# = \rho_i^{-1}\varphi_4 \quad \text{and} \quad b_3^\#(\rho_i E_-^\#)^{-1} = \rho_i^{-1}\varphi_3$$

belong to $L_2^{p\times p}\cap \mathcal{N}_+^{p\times p}$. Thus, the Smirnov maximum principle guarantees that

$$\rho_i^{-1}\varphi_3 \in H_2^{p\times p} \quad \text{and} \quad \rho_i^{-1}\varphi_4 \in H_2^{p\times p} \tag{5.3}$$

and hence that $b_3^\#(\rho_i E_-^\#)^{-1} \in H_2^{p\times p}$ and $(\rho_i E_+)^{-1} b_4^\# \in H_2^{p\times p}$. But this implies the assertion of Step 2, since $(b_3^{(1)})^{-1} b_3 \in \mathcal{E}\cap \mathcal{S}_{\text{in}}^{p\times p}$ and $b_4(b_4^{(1)})^{-1} \in \mathcal{E}\cap\mathcal{S}_{\text{in}}^{p\times p}$.

3. *Verification of* (1). The proof is based on the fact that

$$f\in \mathcal{B}(\mathfrak{E}) \Longleftrightarrow E_+^{-1} f\in H_2^p \quad \text{and} \quad E_-^{-1} f \in (H_2^p)^\perp.$$

In view of Steps 1 and 2, it is easily seen that

$$E_+^{-1} k_\omega^{b_3^{(1)}}\xi = (\rho_i E_+)^{-1}\rho_i k_\omega^{b_3^{(1)}}\xi \in H_2^p \tag{5.4}$$

for every choice of $\omega\in\mathbb{C}$ and $\xi\in\mathbb{C}^p$. Moreover, by a self-evident variant of Step 1 and Step 2,

$$E_-^{-1} k_\omega^{b_3^{(1)}}\xi = \varphi_3^\# b_3^{-1} k_\omega^{b_3^{(1)}}\xi = \{\rho_{-i}^{-1}\varphi_3^\#\}\,\{\rho_{-i} b_3^{-1} k_\omega^{b_3^{(1)}}\xi\} \in (H_2^p)^\perp$$

for every $\omega\in\mathbb{C}$ and $\xi\in\mathbb{C}^p$ (keep (5.3) in mind). This completes the proof of the first assertion in (1); the proof of the second is similar.

4. *Verification of* (2). We shall assume that $b_3^{(1)}(0) = b_4^{(1)}(0) = I_p$. In view of Step 1, $\mathcal{L}$ is a closed subspace of $\mathcal{B}(\mathfrak{E})$ and hence is automatically isometrically included in $\mathcal{B}(\mathfrak{E})$. Moreover, since

$$k_\omega^{b_3^{(1)}}(\lambda) = \frac{I_p - b_3^{(1)}(\lambda) b_3^{(1)}(\omega)^*}{\rho_\omega(\lambda)} = \frac{1}{2\pi i}(R_{\overline{\omega}} b_3^{(1)})(\lambda) b_3^{(1)}(\omega)^*,$$

the resolvent identity $R_\alpha - R_\beta = (\alpha-\beta)R_\alpha R_\beta$ with $\alpha = 0$ and $\beta = \overline{\omega}$ implies that

$$\begin{aligned}(R_0 k_\omega^{b_3^{(1)}})(\lambda) &= \frac{i}{2\pi\overline{\omega}}\{(R_0 b_3^{(1)})(\lambda) - (R_{\overline{\omega}} b_3^{(1)})(\lambda)\} b_3^{(1)}(\omega)^* \\ &= \frac{k_0^{b_3^{(1)}}(\lambda) b_3^{(1)}(\omega)^* - k_\omega^{b_3^{(1)}}(\lambda)}{-\overline{\omega}} \quad \text{if } \omega\neq 0,\end{aligned}$$

and hence that R_0 maps finite sums of the form $\sum k_{\omega_j}^{b_3^{(1)}}\xi_j$ (with $\omega_j\neq 0$) into finite sums of the same form. Similarly, since

$$\ell_\omega^{b_4^{(1)}}(\lambda) = \frac{(b_4^{(1)})^\#(\lambda)(b_4^{(1)})^\#(\omega)^* - I_p}{\rho_\omega(\lambda)} = -\frac{1}{2\pi i}(R_{\overline{\omega}}(b_4^{(1)})^\#)(\lambda)(b_4^{(1)})^\#(\omega)^*$$

$$\begin{aligned} R_0 \ell_\omega^{b_4^{(1)}} &= -\frac{1}{2\pi i}(R_0 R_{\overline{\omega}}(b_4^{(1)})^\#)(\lambda)(b_4^{(1)})^\#(\omega)^* \\ &= -\frac{1}{2\pi i\overline{\omega}}\left\{(R_{\overline{\omega}}(b_4^{(1)})^\#)(\lambda) - (R_0(b_4^{(1)})^\#)(\lambda)\right\} b_4^{(1)}(\overline{\omega}),\end{aligned}$$

R_0 maps finite sums of the form $\sum \ell_{\omega_j}^{b_4^{(1)}} \eta_j$ (with $\omega_j \neq 0$) into finite sums of the same form. Thus, as such sums are dense in $\mathcal{L}$ and R_0 is a bounded operator on $\mathcal{B}(\mathfrak{E})$, $\mathcal{L}$ is invariant under R_0. Therefore, by Theorem 5.2, $\mathcal{L} = \mathcal{B}(\mathfrak{E}_1)$ for some dB matrix $\mathfrak{E}_1 \in \mathcal{I}(j_p)$.

5. *Verification of* (3). Since the blocks in $\mathfrak{E}_1 = \begin{bmatrix} E_-^{(1)} & E_+^{(1)} \end{bmatrix}$ admit factorizations of the form

$$E_-^{(1)} = \widetilde{b}_3 \widetilde{\varphi}_3^{\#} \quad \text{and} \quad E_+^{(1)} = (\widetilde{b}_4)^{-1} \widetilde{\varphi}_4 \quad \text{with } \widetilde{\varphi}_3, \widetilde{\varphi}_4 \in \mathcal{N}_{\text{out}}^{p \times p}$$

and $k_\omega^{b_3^{(1)}} \xi$, $\ell_\omega^{b_4^{(1)}} \xi \in \mathcal{B}(\mathfrak{E}_1)$ for every choice of $\omega \in \mathbb{C}$ and $\xi \in \mathbb{C}^p$,

$$(E_+^{(1)})^{-1} \ell_\omega^{b_4^{(1)}} \xi = (\widetilde{\varphi}_4)^{-1} \widetilde{b}_4 \ell_\omega^{b_4^{(1)}} \xi \in H_2^p \subset \mathcal{N}_+^{p \times p} \tag{5.5}$$

and

$$(E_-^{(1)})^{-1} k_\omega^{b_3^{(1)}} \xi = (\widetilde{\varphi}_3)^{-\#} (\widetilde{b}_3)^{-1} k_\omega^{b_3^{(1)}} \xi \in (H_2^p)^\perp \tag{5.6}$$

for every $\omega \in \mathbb{C}$ and $\xi \in \mathbb{C}^p$. In particular, (5.5) implies that $\widetilde{b}_4 \ell_\omega^{b_4^{(1)}} \xi \in L_2^p \cap \mathcal{N}_+^p = H_2^p$ and hence that

$$\ell_\omega^{b_4^{(1)}} \xi \in (\widetilde{b}_4^{\#} H_2^p) \cap (H_2^p)^\perp = \mathcal{H}_*(\widetilde{b}_4)$$

and, by a similar argument based on (5.6), that

$$k_\omega^{b_3^{(1)}} \xi \in \mathcal{H}(\widetilde{b}_3) \quad \text{for every } \omega \in \mathbb{C} \text{ and } \xi \in \mathbb{C}^p.$$

Thus,

$$\ell_\omega^{b_4^{(1)}}(\omega) \preceq \ell_\omega^{\widetilde{b}_4}(\omega) \quad \text{and} \quad k_\omega^{b_3^{(1)}}(\omega) \preceq k_\omega^{\widetilde{b}_3}(\omega) \quad \text{for every point } \omega \in \mathbb{C}.$$

But this in turn leads easily to the inequalities

$$(b_4^{(1)})^{\#}(\omega)(b_4^{(1)})^{\#}(\omega)^* \preceq \widetilde{b}_4^{\#}(\omega)\widetilde{b}_4^{\#}(\omega)^* \quad \text{and} \quad \widetilde{b}_3(\omega)\widetilde{b}_3(\omega)^* \preceq b_3^{(1)}(\omega) b_3^{(1)}(\omega)^*$$

for $\omega \in \mathbb{C}_+$. Therefore,

$$\widetilde{b}_4 (b_4^{(1)})^{-1} \in \mathcal{S}_{\text{in}}^{p \times p} \quad \text{and} \quad (b_3^{(1)})^{-1} \widetilde{b}_3 \in \mathcal{S}_{\text{in}}^{p \times p}. \tag{5.7}$$

In view of Theorem 5.2, there exists an essentially unique dB matrix $\mathfrak{E}_2 \in \mathcal{I}_R(j_p)$ such that $\mathcal{B}(\mathfrak{E}_2) \subseteq \mathcal{B}(\mathfrak{E}_1)$ isometrically, $\{b_3^{(1)}, b_4^{(1)}\} \in ap(\mathfrak{E}_2)$ and $\mathcal{B}(\mathfrak{E}_2) \subseteq \mathcal{B}(\mathfrak{E})$ isometrically. Therefore, by (2) applied to $\mathfrak{E}_2$ instead of $\mathfrak{E}$, it follows that $\mathcal{B}(\mathfrak{E}_1) \subseteq \mathcal{B}(\mathfrak{E}_2)$ isometrically. Therefore, $\mathcal{B}(\mathfrak{E}_2) = \mathcal{B}(\mathfrak{E}_1)$, as needed.

6. *Verification of* (4). This follows from (2) and (3) by setting $b_3^{(1)} = b_3$ and $b_4^{(1)} = b_4$, since in this case $\mathfrak{E} \in \mathcal{I}_R(j_p)$ if and only if $\mathcal{B}(\mathfrak{E}_1) = \mathcal{B}(\mathfrak{E})$. □

6. The class of strongly regular entire de Branges matrices

A mvf $\mathfrak{E} \in \mathcal{I}(j_p)$ will be called a **strongly regular entire dB matrix** if $\mathcal{B}(\mathfrak{E}) \subset L_2^p$ and there exist a pair of constants $\gamma_1 > 0$ and $\gamma_2 \geq \gamma_1$ such that

$$\gamma_1 \|f\|_{\rm st} \leq \|f\|_{\mathcal{B}(\mathfrak{E})} \leq \gamma_2 \|f\|_{\rm st} \quad \text{for every } f \in \mathcal{B}(\mathfrak{E}). \tag{6.1}$$

The class of strongly regular entire dB matrices will be denoted $\mathcal{I}_{sR}(j_p)$.

Lemma 6.1. *If $\mathfrak{E} \in \mathcal{I}(j_p)$, $\{b_3, b_4\} \in ap(\mathfrak{E})$ and $f \in \mathcal{B}(\mathfrak{E}) \cap L_2^p$, then*

$$f \in \mathcal{H}_*(b_4) \oplus \mathcal{H}(b_3). \tag{6.2}$$

Moreover, $\mathfrak{E} \in \mathcal{I}_{sR}(j_p)$ if and only if

$$\mathcal{B}(\mathfrak{E}) \sim \mathcal{H}_*(b_4) \oplus \mathcal{H}(b_3). \tag{6.3}$$

Proof. If $f \in \mathcal{B}(\mathfrak{E})$, then $E_+^{-1} f \in H_2^p$ and $E_-^{-1} f \in (H_2^p)^\perp$. Thus, in view of the factorizations in (4.12), there exist a pair of mvf's φ_3, $\varphi_4 \in \mathcal{N}_{\rm out}^{p\times p}$ such that

$$\varphi_4 b_4 f \in H_2^p \quad \text{and} \quad \varphi_3^\# b_3^\# f \in (H_2^p)^\perp.$$

Therefore,

$$b_4 f \in \mathcal{N}_+^{p\times 1} \quad \text{and} \quad (b_3^\# f)^\# \in \mathcal{N}_+^{1\times p}.$$

Under the extra assumption that $f \in L_2^p$ (as well as to $\mathcal{B}(\mathfrak{E})$) it follows from the Smirnov maximum principle that $b_4 f \in H_2^p$ and $b_3^\# f \in (H_2^p)^\perp$. Therefore, f is orthogonal to $b_4^\# (H_2^p)^\perp$ and to $b_3 H_2^p$, i.e., (6.2) holds.

Suppose next that $\mathfrak{E} \in \mathcal{I}_{sR}(j_p)$. Then $\mathcal{B}(\mathfrak{E}) \subset L_2^p$ and hence

$$\mathcal{B}(\mathfrak{E}) \subseteq \mathcal{H}_*(b_4) \oplus \mathcal{H}(b_3),$$

since the inclusion (6.2) is in force. At the same time, Theorem 5.3 guarantees that

$$\{k_\alpha^{b_3}\xi + \ell_\beta^{b_4}\eta : \alpha, \beta \in \mathbb{C} \text{ and } \xi, \eta \in \mathbb{C}^p\} \subseteq \mathcal{B}(\mathfrak{E}). \tag{6.4}$$

Let $f \in \mathcal{H}_*(b_4) \oplus \mathcal{H}(b_3)$. Then, since

$$\mathcal{H}_*(b_4) \oplus \mathcal{H}(b_3) = \text{cls}\{k_\alpha^{b_3}\xi + \ell_\beta^{b_4}\eta : \alpha, \beta \in \mathbb{C} \text{ and } \xi, \eta \in \mathbb{C}^p\} \quad \text{in } L_2^p,$$

there exists a Cauchy sequence $\{f_n\}$ of finite linear combinations of vvf's in the set on the left in (6.4) such that $\|f_n - f\|_{\rm st} \to 0$ as $n \uparrow \infty$. In view of (6.1), $\{f_n\}$ is also a Cauchy sequence in $\mathcal{B}(\mathfrak{E})$. Therefore, $\|f_n - g\|_{\mathcal{B}(\mathfrak{E})} \to 0$ as $n \uparrow \infty$ for some vvf $g \in \mathcal{B}(\mathfrak{E})$. But $f = g$, since norm convergence implies pointwise convergence in a RKHS. Thus, (6.3) holds when $\mathfrak{E} \in \mathcal{I}_{sR}(j_p)$.

The converse implication is immediate from the definition of the class $\mathfrak{E} \in \mathcal{I}_{sR}(j_p)$. □

Lemma 6.2. *$\mathcal{I}_{sR}(j_p) \subset \mathcal{I}_R(j_p)$ and the inclusion is proper.*

Proof. If $\mathfrak{E} \in \mathcal{I}_{sR}(j_p)$ and $\{b_3, b_4\} \in ap(\mathfrak{E})$, then (6.3) holds. Assertion (4) of Theorem 5.3 and the relation

$$\mathcal{H}_*(b_4) \oplus \mathcal{H}(b_3) = \text{cls}\{k_\alpha^{b_3}\xi + \ell_\beta^{b_4}\eta : \alpha, \beta \in \mathbb{C} \text{ and } \xi, \eta \in \mathbb{C}^p\} \quad \text{in } L_2^p$$

imply that $\mathfrak{E} \in \mathcal{I}_R(j_p)$. Thus, $\mathcal{I}_{sR}(j_p) \subseteq \mathcal{I}_R(j_p)$. However, the inclusion is proper because the mvf

$$\mathfrak{E}_1(\lambda) = \begin{bmatrix}(1+i\lambda)b_3(\lambda) & (1-i\lambda)b_4(\lambda)^{-1}\end{bmatrix}$$

belongs to the class $\mathcal{I}_R(j_p)$ but not to the class $\mathcal{I}_{sR}(j_p)$. (This will be discussed in more detail in[6].) □

Theorem 6.3. *If $\mathfrak{E} \in \mathcal{I}_{sR}(j_p)$, $\mathfrak{E}_1 \in \mathcal{I}(j_p)$, $\mathcal{B}(\mathfrak{E}_1)$ is a closed subspace of $\mathcal{B}(\mathfrak{E})$ and the inclusion $\mathcal{B}(\mathfrak{E}_1) \subseteq \mathcal{B}(\mathfrak{E})$ is contractive, then $\mathfrak{E}_1 \in \mathcal{I}_{sR}(j_p)$.*

Proof. Let $\{b_3^{(1)}, b_4^{(1)}\} \in ap(\mathfrak{E}_1)$. Then, since $\mathcal{B}(\mathfrak{E}_1) \subseteq \mathcal{B}(\mathfrak{E})$ and $\mathcal{B}(\mathfrak{E}) \subset L_2^p$, Lemma 6.1 ensures that $\mathcal{B}(\mathfrak{E}_1) \subseteq \mathcal{H}_*(b_4^{(1)}) \oplus \mathcal{H}(b_3^{(1)})$.

On the other hand, if $f \in \mathcal{H}_*(b_4^{(1)}) \oplus \mathcal{H}(b_3^{(1)})$, then there exists a sequence $\{f_n\}$, $n = 1, 2, \ldots$ of linear combinations of vvf's of the form $k_\alpha^{b_3^{(1)}} \xi + \ell_\beta^{b_4^{(1)}} \eta$ with $\xi, \eta \in \mathbb{C}^p$ and $\alpha, \beta \in \mathbb{C}$ such that $f_n \to f$ in L_2^p as $n \uparrow \infty$. In view of Theorem 5.3, $f_n \in \mathcal{B}(\mathfrak{E}_1)$ and hence, as $\mathcal{B}(\mathfrak{E}_1) \subseteq \mathcal{B}(\mathfrak{E})$, $f_n \in \mathcal{B}(\mathfrak{E})$ for $n = 1, 2, \ldots$. Thus, as $\mathfrak{E} \in \mathcal{I}_{sR}(j_p)$, $f \in \mathcal{B}(\mathfrak{E})$ and hence, as $\mathcal{B}(\mathfrak{E}_1)$ is closed in $\mathcal{B}(\mathfrak{E})$, $f \in \mathcal{B}(\mathfrak{E}_1)$. Consequently, $\mathcal{B}(\mathfrak{E}_1) = \mathcal{H}_*(b_4^{(1)}) \oplus \mathcal{H}(b_3)$ as vector spaces. Since $\mathfrak{E} \in \mathcal{I}_{sR}(j_p)$ and the inclusion $\mathcal{B}(\mathfrak{E}_1) \subseteq \mathcal{B}(\mathfrak{E})$ is contractive, there exists a constant $\gamma_1 > 0$ such that

$$\gamma_1 \|f\|_{st} \leq \|f\|_{\mathcal{B}(\mathfrak{E})} \leq \|f\|_{\mathcal{B}(\mathfrak{E}_1)} \quad \text{for every } f \in \mathcal{B}(\mathfrak{E}_1).$$

Thus, the identity map T from $\mathcal{B}(\mathfrak{E}_1)$ onto $\mathcal{H}_*(b_4^{(1)}) \oplus \mathcal{H}(b_3^{(1)})$ is subject to the bounds

$$\|Tf\|_{st} = \|f\|_{st} \leq \gamma_1^{-1} \|f\|_{\mathcal{B}(\mathfrak{E}_1)}.$$

Therefore, as T is invertible, a well-known theorem of Banach guarantees that T^{-1} is also bounded, i.e., there exists a constant $\gamma > 0$ such that

$$\|f\|_{\mathcal{B}(\mathfrak{E}_1)} = \|T^{-1} f\|_{\mathcal{B}(\mathfrak{E}_1)} \leq \gamma \|f\|_{st} \quad \text{for every } f \in \mathcal{B}(\mathfrak{E}_1).$$

Consequently, $\mathfrak{E}_1 \in \mathcal{I}_{sR}(j_p)$. □

7. Other directions

In this paper we have not discussed spectral functions nor the role of vector-valued dB spaces as a model for symmetric operators; see e.g., [3] for the former and [12] and the references cited therein for the latter.

The theory of dB spaces $\mathcal{B}(\mathfrak{E})$ based on entire dB matrices $\mathfrak{E}$ is intimately connected with the theory of entire J_p-inner mvfs A and the corresponding RKHS's $\mathcal{H}(A)$.

Let $\mathcal{E} \cap \mathcal{U}(J_p)$ denote the class of $m \times m$ entire J_p-inner mvf's $A(\lambda)$, i.e., entire mvf's such that

$$J_p - A(\lambda) J_p A(\lambda)^* \succeq 0 \quad \text{for } \lambda \in \overline{\mathbb{C}_+} \text{ with equality on } \mathbb{R}.$$

If $A \in \mathcal{E} \cap \mathcal{U}(J_p)$, then the kernel

$$K_\omega^A(\lambda) = \frac{J_p - A(\lambda)J_pA(\omega)^*}{\rho_\omega(\lambda)} \quad \text{for } \lambda \neq \overline{\omega}$$

is positive in the sense of (P3) in Section 3. Therefore, $K_\omega^A(\lambda)$ can be identified as the RK of a RKHS of $2p \times 1$ vvf's that will be denoted $\mathcal{H}(A)$. Moreover, every vvf $f \in \mathcal{H}(A)$ is entire, the mvf

$$\mathfrak{E}(\lambda) = \begin{bmatrix} E_-(\lambda) & E_+(\lambda) \end{bmatrix} = \mathfrak{E}_A(\lambda) \stackrel{\text{def}}{=} \sqrt{2}\begin{bmatrix} 0 & I_p \end{bmatrix} A(\lambda)\mathfrak{V} \tag{7.1}$$

is a dB matrix and

$$K_\omega^{\mathfrak{E}}(\lambda) = \sqrt{2}\begin{bmatrix} 0 & I_p \end{bmatrix} \frac{J_p - A(\lambda)J_pA(\omega)^*}{\rho_\omega(\lambda)} \sqrt{2}\begin{bmatrix} 0 \\ I_p \end{bmatrix}.$$

Furthermore,

$$\mathfrak{E} \in \mathcal{I}(j_p) \quad \text{if and only if} \quad \mathfrak{E} = \mathfrak{E}_A \text{ for some } A \in \mathcal{E} \cap \mathcal{U}(J_p). \tag{7.2}$$

We shall say that a mvf $A \in \mathcal{E} \cap \mathcal{U}(J_p)$ belongs to the class

$$\begin{aligned} \mathcal{E} \cap \mathcal{U}_{rR}(J_p) &\quad \text{if } \mathcal{H}(A) \cap L_2^{m\times m} \text{ is dense in } \mathcal{H}(A), \\ \mathcal{E} \cap \mathcal{U}_{rsR}(J_p) &\quad \text{if } \mathcal{H}(A) \subset L_2^{m\times m}, \\ \mathcal{E} \cap \mathcal{U}_{S}(J_p) &\quad \text{if } \mathcal{H}(A) \cap L_2^{m\times m} = \{0\}, \\ \mathcal{E} \cap \mathcal{U}_{AR}(J_p) &\quad \text{if } A_1 \in \mathcal{E} \cap \mathcal{U}(J_p) \text{ and } A_1^{-1}A \in \mathcal{E} \cap \mathcal{U}(J_p), \text{ then } A_1 \in \mathcal{U}_{rR}(J_p). \end{aligned}$$

A number of other characterizations of these classes are presented in [2].

If $A \in \mathcal{E} \cap \mathcal{U}(J_p)$ is written in block form $A = \begin{bmatrix} a_{ij}(\lambda) \end{bmatrix}$ with $p \times p$ blocks $a_{ij}(\lambda)$ for $i, j = 1, 2$, then the limit

$$\beta = \lim_{\nu \uparrow \infty} \frac{1}{\nu} \Re(a_{11}(i\nu) + a_{12}(i\nu))(a_{21}(i\nu) + a_{22}(i\nu))^{-1}$$

exists and $\beta \succeq 0$. A mvf $A \in \mathcal{E} \cap \mathcal{U}(J_p)$ is said to be **perfect** if $\beta = 0$.

If $\mathfrak{E} \in \mathcal{I}(j_p)$, then there exists a perfect mvf $A \in \mathcal{E} \cap \mathcal{U}(J_p)$ such that $\mathfrak{E} = \mathfrak{E}_A$. Moreover, in this case the operator

$$U_2 : f \in \mathcal{H}(A) \mapsto \sqrt{2}\begin{bmatrix} 0 & I_p \end{bmatrix} f \in \mathcal{B}(\mathfrak{E})$$

is unitary.

It also turns out that

(1) $\mathfrak{E} \in \mathcal{I}(j_p) \iff \mathfrak{E} = \mathfrak{E}_A \quad$ for a perfect mvf $A \in \mathcal{E} \cap \mathcal{U}(J_p)$

and, if $\mathfrak{E} = \mathfrak{E}_A$ for some perfect mvf $A \in \mathcal{E} \cap \mathcal{U}(J_p)$, then

(2) $\mathfrak{E}_A \in \mathcal{I}_S(j_p) \iff A \in \mathcal{E} \cap \mathcal{U}_S(J_p)$,
(3) $\mathfrak{E}_A \in \mathcal{I}_R(j_p) \iff A \in \mathcal{E} \cap \mathcal{U}_{rR}(J_p)$,
(4) $\mathfrak{E}_A \in \mathcal{I}_{sR}(j_p) \Longleftarrow A \in \mathcal{E} \cap \mathcal{U}_{rsR}(J_p)$,

These connections will be discussed in detail in [6].

References

[1] Damir Z. Arov, Regular J-inner matrix-functions and related continuation problems, in: *Linear operators in function spaces*, Oper. Theory Adv. Appl., Vol. **43**, Birkhäuser, Basel, 1990, pp. 63–87.

[2] Damir Z. Arov and Harry Dym, *J-Contractive Matrix Valued Functions and Related Topics*, Cambridge University Press, Cambridge, 2008.

[3] Damir Z. Arov and Harry Dym, *Bitangential Direct and Inverse Problems for Systems of Integral and Differential Equations*, Cambridge University Press, Cambridge, 2012.

[4] Damir Z. Arov and Harry Dym, de Branges spaces of vector valued functions, in *Operator Theory* (ed. Daniel Alpay), Springer, Berlin, 2015, pp. 721–752.

[5] Damir Z. Arov and Harry Dym, Applications of de Branges spaces of vector valued functions, in *Operator Theory* (ed. Daniel Alpay), Springer, Berlin, 2015, pp. 753–776.

[6] Damir Z. Arov and Harry Dym, *Multivariate prediction, de Branges spaces and related extension and inverse problems*, in preparation.

[7] Louis de Branges, Some Hilbert spaces of analytic functions, I., Trans. Amer. Math. Soc. **106** (1963), 445–468.

[8] Louis de Branges, Some Hilbert spaces of analytic functions, II., J. Math. Anal. Appl. **11** (1965), 44–72.

[9] Louis de Branges, Some Hilbert spaces of analytic functions, III., J. Math. Anal. Appl. **12** (1965), 149–186.

[10] Louis de Branges, *Hilbert Spaces of Entire Functions*, Prentice Hall, Englewood Cliffs, N.J., 1968.

[11] Louis de Branges, The expansion theorem for Hilbert spaces of entire functions, in: *Entire Functions and Related Parts of Analysis* (Proc. Sympos. Pure Math., La Jolla, Calif., 1966), Amer. Math. Soc., Providence, R.I., pp. 79–148.

[12] Harry Dym and Santanu Sarkar, Multiplication operatprs with deficiency indices (p, p) and sampling formulas in reproducing kernel Hilbert spaces of entire vector valued functions, preprint.

Damir Z. Arov
Division of Applied Mathematics and Informatics
Institute of Physics and Mathematics
South-Ukrainian National Pedagogical University
65020 Odessa
Ukraine
e-mail: damir.arov@gmail.com

Harry Dym
Department of Mathematics
The Weizmann Institute of Science
Rehovot 761001
Israel
e-mail: harry.dym@weizmann.ac.il

Divided Differences and Two-sided Polynomial Interpolation Over Quaternions

Vladimir Bolotnikov

Dedicated to Professor Daniel Alpay on the occasion of his 60th birthday

Abstract. We consider a two-sided interpolation problem of Lagrange–Hermite type for polynomials over quaternions. Necessary and sufficient condition for the problem to have a solution is given and a particular low-degree solution is constructed in terms of a certain Sylvester equation.

1. Introduction

Let $\mathbb{H}$ be the skew field of real quaternions with imaginary units $\mathbf{i}, \mathbf{j}, \mathbf{k}$ commuting with $\mathbb{R}$ and satisfying $\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = \mathbf{ijk} = -1$. For $\alpha = x_0 + \mathbf{i}x_1 + \mathbf{j}x_2 + \mathbf{k}x_3$ ($x_i \in \mathbb{R}$), its real and imaginary parts, the quaternionic conjugate and the absolute value are defined as $\Re(\alpha) = x_0$, $\Im(\alpha) = \mathbf{i}x_1 + \mathbf{j}x_2 + \mathbf{k}x_3$, $\overline{\alpha} = \Re(\alpha) - \Im(\alpha)$ and $|\alpha| = \sqrt{\alpha\overline{\alpha}} = \sqrt{|\Re(\alpha)|^2 + |\Im(\alpha)|^2}$.

Let $\mathbb{H}[z]$ denote the ring of polynomials in one formal variable z which commutes with quaternionic coefficients. The division algorithm holds in $\mathbb{H}[z]$ on either side, and hence, any (left or right) ideal in $\mathbb{H}[z]$ is principal. We denote by $\langle G \rangle_{\mathbf{r}}$ and $\langle G \rangle_{\boldsymbol{\ell}}$ the right and left ideals generated by $G \in \mathbb{H}[z]$ dropping the subscript if the ideal is two-sided. To exclude non-uniqueness, all generators will be assumed to be monic. We recall that the ring $\mathbb{R}[z]$ of polynomials with real coefficients is the center of $\mathbb{H}[z]$ and that any two-sided ideal in $\mathbb{H}[z]$ is generated by an element from $\mathbb{R}[z]$.

Adapting the main concept from [6] to the current single-variable noncommutative setting, let us say that a finite collection $\{\Phi_i\}_{i=1}^n$ of left (right) evaluation functionals is a *left (right) ideal interpolation scheme* if

1. the solution set of the interpolation problem with homogeneous conditions $\Phi_i(f) = 0$ for $i = 1, \dots, n$ is a right (left) ideal $\mathbb{I} \subset \mathbb{H}[z]$, and
2. any non-homogeneous problem has a unique solution f_0 modulo $\mathbb{I}$.

F. Colombo et al. (eds.), *Advances in Complex Analysis and Operator Theory*,
Trends in Mathematics, DOI 10.1007/978-3-319-62362-7_2

The basic example of the ideal interpolation scheme in $\mathbb{H}[z]$ is the Lagrange left (or right) interpolation problem with interpolation nodes such that none three of them belong to the same similarity (conjugacy) class [7, 8, 14, 15]. To solve a specific problem in $\mathbb{H}[z]$ based on an ideal interpolation scheme, it suffices to find the generator G for the solution set of the homogeneous problem (this usually amounts to computing the least left/right common multiple of several given polynomials) and a particular (low-degree) solution f_0 (which can be done, e.g., using Vandermonde or confluent Vandermonde matrices). Then, by linearity, the solution set for the original problem can be written as $f_0 + \langle G\rangle_{\mathbf{r}}$ (or as $f_0 + \langle G\rangle_{\boldsymbol{\ell}}$) for the left (right) interpolation scheme. In this paper, we make the next step considering the combination of left and right interpolation schemes. Disregarding specific evaluation functionals we may start directly with solution sets for the one-sided parts of the problem. In other words, we will be concerned with the following problem:

(**P**): *Given polynomials the G, H, $f_{\boldsymbol{\ell}}$, $f_{\mathbf{r}} \in \mathbb{H}[z]$, find a polynomial $f \in \mathbb{H}[z]$ such that*

$$f = f_{\boldsymbol{\ell}} + Gp \quad \textit{and} \quad f = f_{\mathbf{r}} + \widetilde{p}H \quad \textit{for some} \quad p, \widetilde{p} \in \mathbb{H}[z]. \tag{1.1}$$

The solvability criterion and a recipe for constructing a particular solution of the problem (**P**) by means of certain Sylvester equation are given in Theorem 1.1 below. In what follows, we use the notation

$$E_n = \begin{bmatrix} 1 & 0 & \dots & 0 \end{bmatrix} \in \mathbb{H}^{1\times n} \quad \text{and} \quad \boldsymbol{\rho}_\alpha(z) := z - \alpha \quad \text{for a fixed } \alpha \in \mathbb{H}. \tag{1.2}$$

Without loss of generality we may (and will) assume that the polynomials G and H are monic and that $\deg f_{\boldsymbol{\ell}} < \deg G$ and $\deg f_{\mathbf{r}} < \deg H$. Since any quaternionic polynomial can be factored into the product of linear factors, we may take G and H in the form

$$G = \boldsymbol{\rho}_{\alpha_1} \cdots \boldsymbol{\rho}_{\alpha_k}, \quad H = \boldsymbol{\rho}_{\beta_m} \cdots \boldsymbol{\rho}_{\beta_1}, \quad \alpha_i, \beta_j \in \mathbb{H}. \tag{1.3}$$

With factorizations (1.3), we associate the matrices

$$\mathcal{J}_{\boldsymbol{\alpha}} = \begin{bmatrix} \alpha_1 & 0 & \dots & 0 \\ 1 & \alpha_2 & \ddots & \vdots \\ & \ddots & \ddots & 0 \\ 0 & & 1 & \alpha_k \end{bmatrix} \quad \text{and} \quad \mathcal{J}_{\boldsymbol{\beta}} = \begin{bmatrix} \beta_1 & 0 & \dots & 0 \\ 1 & \beta_2 & \ddots & \vdots \\ & \ddots & \ddots & 0 \\ 0 & & 1 & \beta_m \end{bmatrix}. \tag{1.4}$$

For a polynomial $g(z) = \sum_{j=0}^N g_j z^j$, let us define the column $\Delta_{\boldsymbol{\ell}}(\boldsymbol{\alpha}; g) \in \mathbb{H}^{k\times 1}$ and the row $\Delta_{\mathbf{r}}(g; \boldsymbol{\beta}) \in \mathbb{H}^{1\times m}$ by the formulas (with E_k and E_m as in (1.2))

$$\Delta_{\boldsymbol{\ell}}(\boldsymbol{\alpha}; g) = \sum_{j=0}^N \mathcal{J}_{\boldsymbol{\alpha}}^j E_k^\top g_j \quad \text{and} \quad \Delta_{\mathbf{r}}(g; \boldsymbol{\beta}) = \sum_{j=0}^N g_j E_m (\mathcal{J}_{\boldsymbol{\beta}}^\top)^j. \tag{1.5}$$

Theorem 1.1. *Given G and H as in* (1.3) *and given polynomials $f_{\boldsymbol{\ell}}$, $f_{\mathbf{r}}$, let $\mathcal{J}_{\boldsymbol{\alpha}}$, $\mathcal{J}_{\boldsymbol{\beta}}$, $\Delta_{\boldsymbol{\ell}}(\boldsymbol{\alpha}; f_{\boldsymbol{\ell}})$ and $\Delta_{\mathbf{r}}(f_{\mathbf{r}}; \boldsymbol{\beta})$ be defined via formulas* (1.4), (1.5). *There is an $f \in \mathbb{H}[z]$*

subject to conditions (1.1) *if and only if the equation*

$$\mathcal{J}_{\boldsymbol{\alpha}} X - X\mathcal{J}_{\boldsymbol{\beta}}^{\top} = \Delta_{\boldsymbol{\ell}}(\boldsymbol{\alpha}; f_{\boldsymbol{\ell}})E_m - E_k^{\top}\Delta_{\mathbf{r}}(f_{\mathbf{r}}; \boldsymbol{\beta}) \tag{1.6}$$

has a solution $X \in \mathbb{H}^{k\times m}$. *If this is the case, then the formula*

$$f_X = f_{\boldsymbol{\ell}} + G\cdot\left(X_{k,1} + X_{k,2}\boldsymbol{\rho}_{\beta_1} + X_{k,3}\boldsymbol{\rho}_{\beta_2}\boldsymbol{\rho}_{\beta_1} + \ldots + X_{k,m}\boldsymbol{\rho}_{\beta_{m-1}}\cdots\boldsymbol{\rho}_{\beta_1}\right) \tag{1.7}$$

establishes a one-to-one correspondence between solutions $X = [X_{i,j}] \in \mathbb{H}^{k\times m}$ *to the equation* (1.6) *and solutions* f_X *to the problem (***P***) of degree less than* $\deg G + \deg H$ *(low-degree solutions).*

Remark 1.2. The entries in the equation (1.6) are based on factorizations (1.3) of G and H which in general are not unique. It follows from Theorem 1.1 that if $G = \boldsymbol{\rho}_{\gamma_1}\cdots\boldsymbol{\rho}_{\gamma_k}$ and $H = \boldsymbol{\rho}_{\eta_m}\cdots\boldsymbol{\rho}_{\eta_1}$ are two other factorizations of G and H, then the equation

$$\mathcal{J}_{\boldsymbol{\gamma}} Y - Y\mathcal{J}_{\boldsymbol{\eta}}^{\top} = \Delta_{\boldsymbol{\ell}}(\boldsymbol{\gamma}; f_{\boldsymbol{\ell}})E_m - E_k^{\top}\Delta_{\mathbf{r}}(f_{\mathbf{r}}; \boldsymbol{\eta}) \tag{1.8}$$

has a solution if and only if the equation (1.6) does. Actually, it can be shown that for matrices $\mathcal{J}_{\boldsymbol{\alpha}}$ and $\mathcal{J}_{\boldsymbol{\gamma}}$ constructed from different factorizations of G, there is an invertible matrix T such that $T\mathcal{J}_{\boldsymbol{\gamma}} = \mathcal{J}_{\boldsymbol{\alpha}}T$ and $TE_k^{\top} = E_k^{\top}$. Similarly, there is an invertible matrix $\widetilde{T}$ such that $\mathcal{J}_{\boldsymbol{\eta}}^{\top}\widetilde{T} = \widetilde{T}\mathcal{J}_{\boldsymbol{\beta}}^{\top}$ and $E_m\widetilde{T} = E_m$. Then it follows from (1.5) that

$$T\Delta_{\boldsymbol{\ell}}(\boldsymbol{\gamma}; g) = \Delta_{\boldsymbol{\ell}}(\boldsymbol{\alpha}; g) \quad \text{and} \quad \Delta_{\mathbf{r}}(g; \boldsymbol{\eta})\widetilde{T} = \Delta_{\mathbf{r}}(g; \boldsymbol{\beta}) \quad \text{for all } g \in \mathbb{H}[z],$$

from which it is seen that Y solves (1.8) if and only if $X = TY\widetilde{T}$ solves (1.6).

The solvability criterion from Theorem 1.1 is fairly satisfactory due to existing procedures verifying whether or not the Sylvester equation (1.6) has a solution. One such procedure is based on the *complex representation* $\varphi(M)$ of a quaternionic matrix M suggested in [17]:

$$\varphi(M) = \begin{bmatrix} M_1 & M_2 \\ -\overline{M}_2 & \overline{M}_1 \end{bmatrix}, \quad \text{where } M = M_1 + M_2\mathbf{j}, \ \ M_1,\, M_2 \in \mathbb{C}^{k\times m}. \tag{1.9}$$

The map $M \mapsto \phi(M)$ is additive and multiplicative and hence, for any solution X of (1.6), the matrix $Y = \varphi(X)$ solves the complex Sylvester equation

$$\varphi(\mathcal{J}_{\boldsymbol{\alpha}})Y - Y\varphi(J_{\boldsymbol{\beta}}^{\top}) = \varphi(\Delta_{\boldsymbol{\ell}}(\boldsymbol{\alpha}; f_{\boldsymbol{\ell}})E_m - E_k^{\top}\Delta_{\mathbf{r}}(f_{\mathbf{r}}; \boldsymbol{\beta})). \tag{1.10}$$

On the other hand, if $Y = \left[\begin{smallmatrix} Y_{11} & Y_{12} \\ Y_{21} & Y_{22}\end{smallmatrix}\right]$ ($Y_{ij} \in \mathbb{C}^{k\times m}$) satisfies (1.10), then the matrix

$$X = \frac{1}{2}\left(Y_{11} + \overline{Y}_{22} + (Y_{12} - \overline{Y}_{21})\,\mathbf{j}\right) \tag{1.11}$$

is a solution to (1.6) (see [13]). Thus, the quaternion Sylvester equation (1.6) has a solution if and only if the complex equation (1.10) does. Moreover, from each solution Y to the equation (1.10), one can combine formulas (1.11) and (1.7) to get a particular solution f to the problem (**P**). The results on complex Sylvester equations are classic and we refer to the survey [16] rather than recall them here. Theorem 1.1 and the complex representation approach are illustrated in Example

3.4 below. We note that combining the formula (1.7) with complex representation (1.9) is particularly efficient in case the problem (**P**) has a unique low-degree solution (see Section 3.1). Otherwise, instead of finding all low-degree solutions via this approach, it seems to be more practical to combine one particular low-degree solution f_X with the general low-degree solution of the homogeneous problem. This second approach will be discussed in Section 3.2. The proof of Theorem 1.1 is presented in Section 2.

2. Divided differences and related Sylvester identity

Divided differences proved to be useful tools in classical polynomial interpolation theory. In the noncommutative setting, divided differences appear in two (left and right) versions which serve to construct low-degree solutions for respectively, left and right Lagrange-Hermite interpolation problems. We will recall these one-sided divided differences in formulas (2.6), (2.7) below. A straightforward computation verifies that for any $\alpha \in \mathbb{H}$ and $f \in \mathbb{H}[z]$,

$$f(z) = f^{\boldsymbol{e_\ell}}(\alpha) + (z-\alpha)\cdot(L_\alpha f)(z) = f^{\boldsymbol{e_r}}(\alpha) + (R_\alpha f)(z)\cdot(z-\alpha), \tag{2.1}$$

where $f^{\boldsymbol{e_\ell}}(\alpha)$ and $f^{\boldsymbol{e_r}}(\alpha)$ are left and right evaluation of f at α given by

$$f^{\boldsymbol{e_\ell}}(\alpha) = \sum_{k=0}^{m} \alpha^k f_k \quad \text{and} \quad f^{\boldsymbol{e_r}}(\alpha) = \sum_{k=0}^{m} f_k \alpha^k \quad \text{if } f(z) = \sum_{j=0}^{m} z^j f_j, \tag{2.2}$$

and where $L_\alpha f$ and $R_\alpha f$ are polynomials of degree $m-1$ given by

$$(L_\alpha f)(z) = \sum_{j+k=0}^{m-1} \alpha^j f_{k+j+1} z^k, \quad (R_\alpha f)(z) = \sum_{j+k=0}^{m-1} f_{k+j+1}\alpha^j z^k. \tag{2.3}$$

Observe that the mappings $f \mapsto L_\alpha f$ and $f \mapsto R_\alpha f$ define a right linear operator L_α and a left linear operator R_α acting on $\mathbb{H}[z]$ (interpreted a a vector space over $\mathbb{H}$). We next observe that for any $\alpha, \beta \in \mathbb{H}$ and $f \in \mathbb{H}[z]$,

$$L_\alpha R_\beta f = R_\beta L_\alpha f \quad \text{and} \quad (L_\alpha f)^{\boldsymbol{e_r}}(\beta) = (R_\beta f)^{\boldsymbol{e_\ell}}(\alpha). \tag{2.4}$$

Making use of (2.3), one may first verify equalities (2.4) for $f(z) = cz^n$ as follows

$$L_\alpha R_\beta f = \sum_{k=0}^{n-2}\Big(\sum_{i+j=n-k-2} \alpha^i c \beta^j\Big) z^k = R_\beta L_\alpha f,$$

$$(L_\alpha f)^{\boldsymbol{e_r}}(\beta) = \sum_{i+j=n-1} \alpha^i c\beta^j = (R_\beta f)^{\boldsymbol{e_\ell}}(\alpha),$$

and then get the general case by linearity. Now we will use operators (2.3) to introduce divided differences.

Given an $f \in \mathbb{H}[z]$, the successive application of the first formula in (2.1) to the elements $\alpha_1, \alpha_2, \ldots \in \mathbb{H}$ and polynomials $f,\ L_{\alpha_1} f,\ L_{\alpha_2} L_{\alpha_1} f, \ldots$ and the second formula in (2.1) to elements $\beta_1, \beta_2, \dots$ and polynomials $f,\ R_{\beta_1}f,\ R_{\beta_2}R_{\beta_1}f, \dots$, lead us, respectively, to representations

$$\begin{aligned} f &= f^{\boldsymbol{e_\ell}}(\alpha_1) + \sum_{j=1}^{\deg f-1} \boldsymbol{\rho}_{\alpha_1}\dots\boldsymbol{\rho}_{\alpha_j} \cdot (L_{\alpha_j}\cdots L_{\alpha_1}f)^{\boldsymbol{e_\ell}}(\alpha_{j+1}), \\ f &= f^{\boldsymbol{e_r}}(\beta_1) + \sum_{j=1}^{\deg f-1} (R_{\beta_j}\cdots R_{\beta_1}f)^{\boldsymbol{e_r}}(\beta_{j+1}) \cdot \boldsymbol{\rho}_{\beta_j}\dots\boldsymbol{\rho}_{\beta_1}, \end{aligned} \tag{2.5}$$

which, being (respectively, left and right) quaternionic analogs of the Newton interpolation formula, suggest to introduce *left* and *right divided differences*

$$\begin{aligned} [\alpha_1; f]_{\boldsymbol{\ell}} &= f^{\boldsymbol{e_\ell}}(\alpha_1), \\ [\alpha_1, \dots, \alpha_i; f]_{\boldsymbol{\ell}} &= (L_{\alpha_{i-1}}\cdots L_{\alpha_1}f)^{\boldsymbol{e_\ell}}(\alpha_i) \quad \text{for} \quad i \ge 2, \end{aligned} \tag{2.6}$$

$$\begin{aligned} [f; \beta_1]_{\mathbf{r}} &= f^{\boldsymbol{e_r}}(\beta_1), \\ [f; \beta_1, \dots, \beta_j]_{\mathbf{r}} &= (R_{\beta_{j-1}}\cdots R_{\beta_1}f)^{\boldsymbol{e_r}}(\beta_j) \quad \text{for} \quad j \ge 2, \end{aligned} \tag{2.7}$$

based on given tuples $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_k)$ and $\boldsymbol{\beta} = (\beta_1, \dots, \beta_m)$.

Remark 2.1. It follows from (2.5)–(2.7) that for G and H of the form (1.3),

1. $f \in \langle G\rangle_{\mathbf{r}}$ if and only if $[\alpha_1, \dots, \alpha_i; f]_{\boldsymbol{\ell}} = 0$ for $i = 1, \dots, k$;
2. $f \in \langle H\rangle_{\boldsymbol{\ell}}$ if and only if $[f; \beta_1, \dots, \beta_j]_{\mathbf{r}} = 0$ for $j = 1, \dots, m$.

It turns out that in the context of a two-sided interpolation problem, one needs "two-sided" divided differences which we will refer to as to *mixed divided differences* and which we define as follows:

$$\begin{aligned} [\alpha_1, \dots, \alpha_i; f; \beta_1, \dots, \beta_j] &:= (L_{\alpha_i}\cdots L_{\alpha_1}R_{\beta_{j-1}}\cdots R_{\beta_1}f)^{\boldsymbol{e_r}}(\beta_j) \\ &= (L_{\alpha_{i-1}}\cdots L_{\alpha_1}R_{\beta_j}\cdots R_{\beta_1}f)^{\boldsymbol{e_\ell}}(\alpha_i), \end{aligned} \tag{2.8}$$

where the second equality holds due to (2.4). For $j = 1$ and for $i = 1$ the first and the second formulas in (2.8) take the form

$$\begin{aligned} [\alpha_1, \dots, \alpha_i; f; \beta_1] &= (L_{\alpha_i}\cdots L_{\alpha_1}f)^{\boldsymbol{e_r}}(\beta_1), \\ [\alpha_1; f; \beta_1, \dots, \beta_j] &= (R_{\beta_j}\cdots R_{\beta_1}f)^{\boldsymbol{e_\ell}}(\alpha_1). \end{aligned} \tag{2.9}$$

In the next lemma, the column $\Delta_{\boldsymbol{\ell}}(\boldsymbol{\alpha}; f)$ and the row $\Delta_{\mathbf{r}}(f; \boldsymbol{\beta})$ defined in (2.10), (2.11) as certain sums (in accordance to (1.5)), turn out to be the column and the row of left and right divided differences of f based on the tuples $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$, respectively. We refer to [8, Remark 2.5] for the proof of the second equalities in (2.10) and (2.11).