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Fabrice Bethuel
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Ginzburg-Landau Vortices

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INTRODUCTION

The original motivation of this study comes from the following questions that were mentioned to one of us by H. Matano.

Let

$$G = B_1 = \{x = (x_1, x_2) \in \mathbf{R}^2 ;\ x_1^2 + x_2^2 = |x|^2 < 1\}.$$

Consider the Ginzburg-Landau functional

$$E_\varepsilon(u) = \frac{1}{2}\int_G |\nabla u|^2 + \frac{1}{4\varepsilon^2}\int_G (|u|^2 - 1)^2 \tag{1}$$

which is defined for maps $u \in H^1(G;\mathbb{C})$ also identified with $H^1(G;\mathbf{R}^2)$.

Fix the boundary condition

$$g(x) = x \quad \text{on} \quad \partial G$$

and set

$$H_g^1 = \{u \in H^1(G;\mathbb{C});\quad u = g \quad \text{on} \quad \partial G\}.$$

It is easy to see that

$$\operatorname*{Min}_{u \in H_g^1} E_\varepsilon(u) \tag{2}$$

is achieved by some u_ε that is smooth and satisfies the Euler equation

$$\begin{cases} -\Delta u_\varepsilon = \dfrac{1}{\varepsilon^2} u_\varepsilon (1 - |u_\varepsilon|^2) & \text{in } G, \\ u_\varepsilon = g & \text{on } \partial G. \end{cases} \tag{3}$$

The maximum principle easily implies (see e.g., F. Bethuel, H. Brezis and F. Hélein [2]) that any solution u_ε of (3) satisfies $|u_\varepsilon| \le 1$ in G. In particular, a subsequence (u_{ε_n}) converges in the $w^\star - L^\infty(G)$ topology to a limit $u^\star$. Clearly, $|u^\star(x)| \le 1$ a.e. It is very easy to prove (see Chapter III) that

$$\int_G (|u_\varepsilon|^2 - 1)^2 \le C\varepsilon^2 |\log \varepsilon| \tag{4}$$

and thus $|u_{\varepsilon_n}(x)| \to 1$ a.e. This suggests that $|u^\star(x)| = 1$ a.e. However, such a claim is not clear at all since we do not know, at this stage, that $u_{\varepsilon_n} \to u^\star$ a.e. It turns out to be true that $|u^\star(x)| = 1$ a.e. — but we have no simple proof. This fact is derived as a consequence of a delicate analysis (see Chapter VI).

The original questions of H. Matano were:

Question 1: Does $\lim_{\varepsilon\to 0} u_\varepsilon(x)$ exist a.e.?

Question 2: What is $u^\star$? Do we have $u^\star(x) = x/|x|$?

Question 3: What can be said about the zeroes of u_ε? If they are isolated do they have degree ± 1 (in the sense of Section IX.1)?

These questions have prompted us to consider a more general setting. Let $G \subset \mathbb{R}^2$ be a smooth, bounded and simply connected domain in $\mathbb{R}^2$. Fix a (smooth) boundary condition $g : \partial G \to S^1$ and consider a minimizer u_ε of problem (2) as above. Our purpose is to study the behavior of u_ε as $\varepsilon \to 0$.

The Brouwer degree

$$d = \deg(g, \partial G) \tag{5}$$

(i.e., the winding number of g considered as a map from ∂G into S^1) plays a crucial role in the asymptotic analysis of u_ε.

Case $\mathbf{d = 0}$. This case is easy because $H^1_g(G;\, S^1) \neq \emptyset$ and thus the minimization problem

$$\operatorname*{Min}_{u\in H^1_g(G;S^1)} \int_G |\nabla u|^2 \tag{6}$$

makes sense. In fact, problem (6) has a unique solution u_0 that is a smooth harmonic map from G into S^1, i.e.,

$$-\Delta u_0 = u_0 |\nabla u_0|^2 \quad \text{in} \quad G.$$

Moreover (see e.g., Lemma 1 in F. Bethuel, H. Brezis and F. Hélein [2])

$$u_0 = e^{i\varphi_0} \quad \text{in} \quad G$$

where φ_0 is a harmonic function (unique mod $2\pi\mathbb{Z}$) such that

$$e^{i\varphi_0} = g \quad \text{on} \quad \partial G.$$

We have proved in F. Bethuel, H. Brezis and F. Hélein [2] (see also Appendix I at the end of the book) that $u_\varepsilon \to u_0$ in $C^{1,\alpha}(\overline{G})$ and in $C^k_{\text{loc}}(G)\ \ \forall k$; in particular,

$$\int_G |\nabla u_\varepsilon|^2 \text{ remains bounded as } \varepsilon \to 0. \tag{7}$$

We have also obtained rates of convergence for $\|u_\varepsilon - u_0\|$ in various norms.

Case d≠0. Throughout the book we assume that $d > 0$ since the case $d < 0$ reduces to the previous case by complex conjugation. Here, the main difficulty stems from the fact that

$$H_g^1(G;S^1) = \emptyset. \tag{8}$$

Indeed, suppose not, and say that $H_g^1(G;S^1) \neq \emptyset$, then we could consider, as above,

$$\operatorname*{Min}_{H_g^1(G;S^1)} \int_G |\nabla u|^2. \tag{9}$$

A minimizer exists and is smooth up to ∂G, e.g., by a result of C. Morrey [1],[2]. In particular, there would be some $u \in C(\overline{G};S^1)$ such that $u = g$ on ∂G. Standard degree theory shows that this is impossible since g can be homotopied in S^1 to a constant. Alternatively, one could also use $H^{1/2}(S^1;S^1)$ degree theory (see a result of L. Boutet de Monvel and O. Gabber quoted in A. Boutet de Monvel-Berthier, V. Georgescu and R. Purice [1], [2]) to show that $H_g^1(G;S^1) = \emptyset$.

In this case, problem (9) does not make sense. In order to get around this topological obstruction we are led to the following idea. Enlarge the class of testing functions to

$$H_g^1(G;\mathbb{C}).$$

(Clearly this set is always nonempty.) But on the other hand, add a penalization in the energy that "forces" $|u|$ to be close to 1. The simplest penalty that comes to mind is

$$\frac{1}{\varepsilon^2}\int_G (|u|^2-1)^2.$$

Therefore, we are led very naturally to

$$\operatorname*{Min}_{H_g^1(G;\mathbb{C})} E_\varepsilon.$$

Here, in contrast with the previous case,

$$\int_G |\nabla u_\varepsilon|^2 \to +\infty, \quad \text{as } \varepsilon \to 0, \tag{10}$$

(otherwise, $u_{\varepsilon_n} \rightharpoonup \tilde{u}$ weakly in H^1 and $u_{\varepsilon_n} \to \tilde{u}$ a.e., so that $|\tilde{u}| = 1$, a.e.; thus $\tilde{u} \in H_g^1(G;S^1)$ — impossible by (8)). However, we may still hope that

$$u_\star(x) = \lim\ u_{\varepsilon_n}(x) \ \text{ exists for a.e. } x \in G$$

(naturally, with $\int_G |\nabla u_\star|^2 = \infty$). If this is indeed the case then $u_\star$ can be viewed as a "generalized solution" of problem (9).

Of course, many other "penalties" can be devised. They all seem to lead to the same class of generalized solutions. For example, one other natural penalty consists of drilling a few little holes $B(a_i, \rho)$ in G and considering the domain $G_\rho = G \setminus \bigcup_i B(a_i, \rho)$. In this case there is no topological obstruction and

$$H^1_g(G_\rho; S^1) \neq \emptyset$$

(we do not impose a Dirichlet condition on $\partial B(a_i, \rho)$). Then, one may consider the problem

$$\operatorname*{Min}_{H^1_g(G_\rho;S^1)} \int_{G_\rho} |\nabla u|^2$$

and analyze what happens as $\rho \to 0$. Here, the points (a_i) are free to move and some configurations will turn out to be "better" than others (see Section I.4 and Chapter VIII).

Going back to a minimizer u_ε of the original functional E_ε, our main results are the following:

Theorem 0.1. *Assume G is starshaped. Then there is a subsequence $\varepsilon_n \to 0$ and exactly d points $a_1, a_2, \dots, a_d$ in G and a smooth harmonic map $u_\star$ from $G \setminus \{a_1, a_2, \dots, a_d\}$ into S^1 with $u_\star = g$ on ∂G such that*

$$u_{\varepsilon_n} \to u_\star \text{ in } C^k_{\text{loc}}(G \setminus \bigcup_i \{a_i\}) \ \forall k \text{ and in } C^{1,\alpha}(\bar{G} \setminus \bigcup_i \{a_i\}) \ \forall \alpha < 1.$$

In addition, each singularity has degree +1 and, more precisely, there are complex constants (α_i) with $|\alpha_i| = 1$ such that

$$\left| u_\star(z) - \alpha_i \frac{(z - a_i)}{|z - a_i|} \right| \leq C|z - a_i|^2 \text{ as } z \to a_i, \ \forall i. \tag{11}$$

This theorem answers, in particular, Question 1 above. In this theorem it is essential (in general) to pass to a subsequence. For example, if G is the unit disc and $g = e^{2i\theta}$ then, for ε small, u_ε is **not** unique and various subsequences converge to different limits (see Section VIII.5). However, in some cases, for example $g(\theta) = e^{i\theta}$, the full sequence (u_ε) converges to a well defined limit (see Section VIII.4).

So far, we have not said anything about the location of the singularities. Our next result tells us where to find them. For this purpose, we introduce, for any given configuration $b = (b_1, b_2, \ldots, b_d)$ of distinct points in G, the function

(12) $$W(b) = -\pi \sum_{i \neq j} \log |b_i - b_j| + \frac{1}{2} \int_{\partial G} \Phi(g \times g_\tau) - \pi \sum_{i=1}^{d} R(b_i)$$

where Φ is the solution of the linear Neumann problem

$$\begin{cases} \Delta\Phi = 2\pi \sum\limits_{i=1}^{d} \delta_{b_i} & \text{in } G, \\ \dfrac{\partial\Phi}{\partial\nu} = g \times g_\tau & \text{on } \partial G, \end{cases}$$

(ν is the outward normal to ∂G and τ is a unit tangent vector to ∂G such that (ν, τ) is direct) and

$$R(x) = \Phi(x) - \sum_{i=1}^{d} \log |x - b_i|.$$

Note that $R \in C(\overline{G})$, so that $R(b_i)$ makes sense.

The function W, called the **renormalized energy**, has the following properties (see Section I.4):

(i) $W \to +\infty$ as two of the points b_i coalesce,

(ii) $W \to +\infty$ as one of the points b_i tends to ∂G (since $R(b_i) \to -\infty$ as $b_i \to \partial G$).

In other words, the singularities b_i **repel** each other, but the boundary condition on ∂G produces a **confinement effect**. In particular W achieved its minimum on G^d and every minimizing configuration consists of d **distinct points** in G^d **(not** $\bar{G}^d$**)**.

The location of the points (a_i) in Theorem 0.1 is governed by W through the following:

Theorem 0.2. *Let (a_i) be as in Theorem 0.1. Then (a_i) is a minimizer for W on G^d.*

The expression W comes up naturally in the following computation. Given **any** configuration $b = (b_1, b_2, \ldots, b_d)$ of distinct points in G, let $G_\rho = G \setminus \bigcup_i B(b_i, \rho)$. Consider the class

(13) $$\mathcal{E}_\rho = \left\{ v \in H^1(G_\rho; S^1) \middle| \begin{matrix} v = g \text{ on } \partial G & \text{and} \\ \deg(v, \partial B(b_i, \rho)) = 1 & \forall i \end{matrix} \right\}.$$

One proves (see Theorem I.2) that there exists a unique minimizer u_ρ for the problem

$$\text{(14)} \qquad \operatorname*{Min}_{u\in\mathcal{E}_\rho} \int_{G_\rho} |\nabla u|^2$$

and that (see Theorem I.7) the following expansion holds:

$$\text{(15)} \qquad \frac{1}{2}\int_{G_\rho} |\nabla u_\rho|^2 = \pi\, d|\log\, \rho| + W(b) + O(\rho) \quad \text{as } \rho \to 0.$$

In other words, W is what remains in the energy after the singular "**core energy**" $\pi\, d|\log\, \rho|$ has been removed. (The idea of removing an infinite core energy is common in physics; see e.g., M. Kléman [1]). Moreover, as $\rho \to 0$, u_ρ converges to some u_0 that has the following properties:

$$\text{(16)} \qquad u_0 \text{ is a smooth harmonic map in } G \setminus \bigcup_i \{b_i\}$$

$$\text{(17)} \qquad u_0 = g \quad \text{on } \partial G$$

$$\text{(18)} \qquad \left| u_0(z) - \beta_i \frac{(z-b_i)}{|z-b_i|} \right| \le C|z-b_i| \quad \text{as } z \to b_i,\ \forall i$$

for some complex numbers β_i with $|\beta_i| = 1\ \forall i$.

In fact, given any configuration $b \in G^d$ of distinct points, there is a unique u_0 satisfying (16), (17) and (18) (see Corollary I.1). We call this u_0 the **canonical harmonic map** associated to the configuration b.

There is an **explicit formula** for u_0 (see Corollary I.2):

$$\text{(19)} \qquad u_0(z) = e^{i\varphi(z)} \frac{(z-b_1)}{|z-b_1|}\frac{(z-b_2)}{|z-b_2|}\cdots\frac{(z-b_d)}{|z-b_d|}$$

where φ is the solution of the Dirichlet problem

$$\text{(20)} \qquad \begin{cases} \Delta\varphi = 0 & \text{in } G \\ \varphi = \varphi_0 & \text{on } \partial G \end{cases}$$

and φ_0 is defined on ∂G by

$$\text{(21)} \qquad e^{i\varphi_0(z)} = g(z)\frac{|z-b_1|}{(z-b_1)}\frac{|z-b_2|}{(z-b_2)}\cdots\frac{|z-b_d|}{(z-b_d)}.$$

(Note that the right-hand side in (21) is a map from ∂G into S^1 of degree zero so that φ_0 is well defined as a single-valued smooth function.)

For a **general configuration** b estimate (18) **cannot** be improved. However, for the **special** configuration as described in Theorem 0.1 we have the better estimate (11). That property, which may be written as

$$\nabla \left(u_\star(x) \frac{|x-a_i|}{(x-a_i)} \right)(a_i) = 0 \quad \forall i, \tag{22}$$

is related to the fact that $a = (a_1, a_2, \dots, a_d)$ is a critical point of W on G^d. It is extremely useful in localizing the singularities of $u_\star$ (see Section VIII.4).

The role of condition (22) has been strongly emphasized (in the case of a single singularity) by J. Neu[1] and by P. Fife and L. Peletier [1]. They show that (22) must be satisfied in order to be able to carry out a matched asymptotic expansion argument for (3).

Equation (22) also bears some resemblance with the results concerning the location of the blow-up points for the problem

$$-\Delta u = u^{p-\varepsilon} \qquad \text{or } -\Delta u = u^p + \varepsilon u \quad \text{in } \Omega \subset \mathbb{R}^n$$

with critical exponent $p = (n+2)/(n-2)$. There, the blow-up points a satisfy

$$\nabla H(a) = 0$$

where H is the regular part of the Green's functions (see H. Brezis and L. Peletier [1] and O. Rey [1], [2]).

To complete the description of $u_\star$ we have:

Theorem 0.3. *Let (a_i) and $u_\star$ be as in Theorem 0.1. Then $u_\star$ is* **the** *canonical harmonic map associated to the configuration*

$$a = (a_1, a_2, \dots, a_d).$$

Conclusion: In general, W may have several minima. However, once the location of a_i is known, then $u_\star$ is completely determined. In some important cases W has a unique minimizer that can be identified explicitly; for example when $G = B_1$ and $g(x) = x$:

Theorem 0.4. *Assume $G = B_1$ and $g(x) = x$. Let u_ε be a minimizer for (1), then, $\forall x \neq 0$,*

$$u_\varepsilon(x) \to u_\star(x) = \frac{x}{|x|} \ \textit{as } \varepsilon \to 0.$$

This answers Question 2 above.

Theorem 0.4 can be viewed as the 2-dimensional analogue of a result of H. Brezis, J. M. Coron and E. Lieb [1], which asserts that the unique minimizer of the problem

$$\operatorname*{Min}_{u\in H^1_g(B^3;S^2)} \int_{B^3} |\nabla u|^2 \quad \text{with } \ g(x)=x$$

is $u(x) = x/|x|$. More generally, F.H. Lin [1] has obtained the same conclusion for the problem

$$\operatorname*{Min}_{u\in H^1_g(B^n;S^{n-1})} \int_{B^n} |\nabla u|^2 \quad \text{for any } \ n\geq 3.$$

Next, we study the zeroes of u_ε. Let us recall some earlier works on that question. It has been proved by C. Elliott, H. Matano and T. Qi [1] that (for every $\varepsilon > 0$) the zeroes of any minimizer u_ε of (2) are isolated. P. Bauman, N. Carlson and D. Phillips [1] have shown, in particular, that if $G = B_1$ and $\deg(g, \partial G) = 1$ with $g(\theta)$ strictly increasing then (for every $\varepsilon > 0$) there is a unique zero of any minimizer u_ε of (2).

Our main result concerning the zeroes of u_ε is the following:

Theorem 0.5. *Let G be a starshaped domain and let $d = \deg(g, \partial G)$. Then, for $\varepsilon < \varepsilon_0$ depending only on g and G, u_ε has exactly d zeroes of degree $+1$.*

Remark 0.1. If $d \geq 2$ we give an example in Section VIII.5 showing that the conclusion of Theorem 0.5 fails when ε is large. The following happens: when ε is large u_ε has a single zero of degree d and, as $\varepsilon \to 0$, this zero splits into d zeroes of degree $+1$.

Finally we analyze the behavior as $\varepsilon \to 0$ of solutions v_ε of the Ginzburg-Landau equation (3), which **need not be minimizers** of E_ε. We prove that some of the results presented above for minimizers still hold for solutions of (3). In particular, v_{ε_n} converges to some limit $v_\star$ in $C^k_{\rm loc}(G\backslash\bigcup_j\{a_j\})$ where $\{a_j\}$ is a finite set. However, by contrast with the previous situation, we have no information about $\operatorname{card}(\bigcup_j\{a_j\})$ and $\deg(v_\star, a_j)$ need **not** be $+1$. More precisely, we have

Theorem 0.6. *Assume G is starshaped. Then there exist a subsequence $\varepsilon_n \to 0$, k points $a_1, a_2, \dots, a_k$ in G and a smooth harmonic map $v_\star : \overline{G}\setminus\bigcup_j\{a_j\} \to S^1$ with $v_\star = g$ on ∂G such that*

$$v_{\varepsilon_n} \to v_\star \ \text{in } C^\ell_{\rm loc}(G\setminus \cup_j\{a_j\})\ \forall \ell \ \text{and in } C^{1,\alpha}(\overline{G}\setminus\cup_j\{a_j\}) \quad \forall\alpha<1.$$

Moreover, there exist integers $d_1, d_2, \ldots, d_k \in \mathbb{Z} \setminus \{0\}$ *and a smooth harmonic function* $\varphi : \overline{G} \to \mathbb{R}$ *such that*

$$v_*(z) = e^{i\varphi(z)} \frac{(z - a_1)^{d_1}}{|z - a_1|^{d_1}} \cdots \frac{(z - a_k)^{d_k}}{|z - a_k|^{d_k}}.$$

In addition, we have

$$\nabla \left(v_*(z) \frac{|z - a_j|^{d_j}}{(z - a_j)^{d_j}} \right) (a_j) = 0 \quad \forall j,$$

which expresses that $(a_j,\, d_j)$ *is a critical point of some appropriate renormalized energy* W.

Remark 0.2. We emphasize that k need not be equal to d. However there is a bound for k in terms of g and G, and similarly for $\sum_j |d_j|$. We also emphasize that Theorem 0.6 is of interest even in the case where $d = \deg(g, \partial\Omega) = 0$ (we recall that the result of F. Bethuel, H. Brezis and F. Hélein [2] concerns only the analysis, as $\varepsilon \to 0$, of **minimizers** of E_ε when $d = 0$).

Analogies in physics. The results discussed in this book present striking analogies to numerous theoretical and experimental discoveries in the area of superconductors and superfluids over the past 40 years. Functionals of the form $E_\varepsilon(u)$ were originally introduced by V. Ginzburg and L. Landau [1] in the study of phase transition problems occurring in superconductivity; similar models are also used in superfluids such as helium II (see V. Ginzburg and L. Pitaevskii [1]) and in XY-magnetism. There is a considerable amount of literature on this huge subject; some of the standard references are: P. G. DeGennes [1], R. Donnelly [1], J. Kosterlitz and D. Thouless [1], D. Nelson [1], P. Nozières and D. Pines [1], R. Parks [1], D. Saint-James, G. Sarma and E. J. Thomas [1], D. Tilley and J. Tilley [1], M. Tinkham [1]. The unknown u represents a complex order parameter (i.e., with two degrees of freedom). In the physics literature u — often denoted ψ — is called a **condensate wave function** or a Higgs field. The parameter ε, which has the **dimension of a length**, depends on the material and its temperature. In the physics literature it is called the **(Ginzburg-Landau) coherence length** (or healing length or **core radius**) and is often denoted by $\xi = \xi(T)$. For temperatures $T < T_c$ (the critical temperature) with T not too close to T_c , $\xi(T)$ is **extremely small,** typically of the order of some hundreds of angstroms in superconductors, and of the order of a few angstroms in superfluids. Hence, it is of interest to study the asymptotics as $\varepsilon \to 0$, even though the limiting problem (at