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Stochastic Calculus of Variations in Mathematical Finance

Springer

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Dedicated to Kiyosi Itô

Preface

Stochastic Calculus of Variations (or Malliavin Calculus) consists, in brief, in constructing and exploiting natural differentiable structures on abstract probability spaces; in other words, Stochastic Calculus of Variations proceeds from a merging of differential calculus and probability theory.

As optimization under a random environment is at the heart of mathematical finance, and as differential calculus is of paramount importance for the search of extrema, it is not surprising that Stochastic Calculus of Variations appears in mathematical finance. The computation of price sensitivities (or Greeks) obviously belongs to the realm of differential calculus.

Nevertheless, Stochastic Calculus of Variations was introduced relatively late in the mathematical finance literature: first in 1991 with the Ocone-Karatzas hedging formula, and soon after that, many other applications appeared in various other branches of mathematical finance; in 1999 a new impetus came from the works of P. L. Lions and his associates.

Our objective has been to write a book with complete mathematical proofs together with a relatively light conceptual load of abstract mathematics; this point of view has the drawback that often theorems are not stated under minimal hypotheses.

To faciliate applications, we emphasize, whenever possible, an approach through finite-dimensional approximation which is crucial for any kind of numerical analysis. More could have been done in numerical developments (calibrations, quantizations, etc.) and perhaps less on the geometrical approach to finance (local market stability, compartmentation by maturities of interest rate models); this bias reflects our personal background.

Chapter 1 and, to some extent, parts of Chap. 2, are the only prerequisites to reading this book; the remaining chapters should be readable independently of each other. Independence of the chapters was intended to facilitate the access to the book; sometimes however it results in closely related material being dispersed over different chapters. We hope that this inconvenience can be compensated by the extensive Index.

The authors wish to thank A. Sulem and the joint Mathematical Finance group of INRIA Rocquencourt, the Université de Marne la Vallée and Ecole Nationale des Ponts et Chaussées for the organization of an International

Symposium on the theme of our book in December 2001 (published in *Mathematical Finance*, January 2003). This Symposium was the starting point for our joint project.

Finally, we are greatly indepted to W. Schachermayer and J. Teichmann for reading a first draft of this book and for their far-reaching suggestions. Last not least, we implore the reader to send any comments on the content of this book, including errors, via email to `thalmaier@math.univ-poitiers.fr`, so that we may include them, with proper credit, in a Web page which will be created for this purpose.

Paris,
April, 2005

Paul Malliavin
Anton Thalmaier

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Gaussian Stochastic Calculus of Variations

The Stochastic Calculus of Variations [141] has excellent basic reference articles or reference books, see for instance [40, 44, 96, 101, 144, 156, 159, 166, 169, 172, 190–193, 207]. The presentation given here will emphasize two aspects: firstly finite-dimensional approximations in view of the finite dimensionality of any set of financial data; secondly numerical constructiveness of divergence operators in view of the necessity to realize fast numerical Monte-Carlo simulations. The second point of view will be enforced through the use of *effective vector fields.*

1.1 Finite-Dimensional Gaussian Spaces, Hermite Expansion

The One-Dimensional Case

Consider the canonical Gaussian probability measure γ_1 on the real line $\mathbb{R}$ which associates to any Borel set A the mass

$$\gamma_1(A) = \frac{1}{\sqrt{2\pi}} \int_A \exp\left(-\frac{\xi^2}{2}\right) d\xi \ . \tag{1.1}$$

We denote by $L^2(\gamma_1)$ the Hilbert space of square-integrable functions on $\mathbb{R}$ with respect to γ_1. The monomials $\{\xi^s : s \in \mathbb{N}\}$ lie in $L^2(\gamma_1)$ and generate a dense subspace (see for instance [144], p. 6).

On dense subsets of $L^2(\gamma_1)$ there are two basic operators: the *derivative* (or *annihilation*) operator $\partial\varphi := \varphi'$ and the *creation* operator $\partial^*\varphi$, defined by

$$(\partial^*\varphi)(\xi) = -(\partial\varphi)(\xi) + \xi\varphi(\xi) \ . \tag{1.2}$$

Integration by parts gives the following duality formula:

$$(\partial\varphi|\psi)_{L^2(\gamma_1)} := \mathbb{E}[(\partial\varphi)\,\psi] = \int_{\mathbb{R}} (\partial\varphi)\,\psi\, d\gamma_1 = \int_{\mathbb{R}} \varphi\,(\partial^*\psi)\, d\gamma_1 = (\varphi|\partial^*\psi)_{L^2(\gamma_1)} \ .$$

Moreover we have the identity

$$\partial\partial^* - \partial^*\partial = 1$$

which is nothing other than the Heisenberg commutation relation; this fact explains the terminology creation, resp. annihilation operator, used in the mathematical physics literature. As the *number operator* is defined as

$$\mathcal{N} = \partial^*\partial\ , \tag{1.3}$$

we have

$$(\mathcal{N}\varphi)(\xi) = -\varphi''(\xi) + \xi\varphi'(\xi)\ .$$

Consider the sequence of Hermite polynomials given by

$$H_n(\xi) = (\partial^*)^n(1), \quad \text{i.e.,}\ \ H_0(\xi) = 1,\ H_1(\xi) = \xi,\ H_2(\xi) = \xi^2 - 1,\ \text{etc.}$$

Obviously H_n is a polynomial of degree n with leading term ξ^n. From the Heisenberg commutation relation we deduce that

$$\partial(\partial^*)^n - (\partial^*)^n\partial = n(\partial^*)^{n-1}.$$

Applying this identity to the constant function 1, we get

$$H_n' = nH_{n-1}, \quad \mathcal{N}H_n = nH_n\,;$$

moreover

$$\mathbb{E}[H_nH_p] = \big((\partial^*)^n1|H_p\big)_{L^2(\gamma_1)} = \big(1|\partial^nH_p\big)_{L^2(\gamma_1)} = \mathbb{E}[\partial^nH_p]\ . \tag{1.4}$$

If $p < n$ the r.h.s. of (1.4) vanishes; for $p = n$ it equals $n!$. Therefore

$$\left\{\frac{1}{\sqrt{n!}}H_n,\ n = 0, 1, \ldots\right\} \quad \text{constitutes an orthonormal basis of } L^2(\gamma_1).$$

Proposition 1.1. *Any C^∞-function φ with all its derivatives $\partial^n\varphi \in L^2(\gamma_1)$ can be represented as*

$$\varphi = \sum_{n=0}^{\infty} \frac{1}{n!}\,\mathbb{E}(\partial^n\varphi)\,H_n\ . \tag{1.5}$$

Proof. Using

$$\mathbb{E}[\partial^n\varphi] = (\partial^n\varphi\,|\,1)_{L^2(\gamma_1)} = (\varphi\,|\,(\partial^*)^n1)_{L^2(\gamma_1)} = \mathbb{E}[\varphi\,H_n],$$

the proof is completed by the fact that the $H_n/\sqrt{n!}$ provide an orthonormal basis of $L^2(\gamma_1)$. □

Corollary 1.2. *We have*

$$\exp\left(c\xi - \frac{1}{2}c^2\right) = \sum_{n=0}^{\infty} \frac{c^n}{n!}\,H_n(\xi), \quad c \in \mathbb{R}.$$

Proof. Apply (1.5) to $\varphi(\xi) := \exp(c\xi - c^2/2)$. □

The d-Dimensional Case

In the sequel, the space $\mathbb{R}^d$ is equipped with the Gaussian product measure $\gamma_d = (\gamma_1)^{\otimes d}$. Points $\xi \in \mathbb{R}^d$ are represented by their coordinates ξ^α in the standard base e_α, $\alpha = 1, \dots, d$. The derivations (or annihilation operators) ∂_α are the partial derivatives in the direction e_α; they constitute a commuting family of operators. The creation operators ∂_α^* are now defined as

$$(\partial_\alpha^* \varphi)(\xi) := -(\partial_\alpha \varphi)(\xi) + \xi^\alpha \varphi(\xi);$$

they constitute a family of commuting operators indexed by α.

Let $\mathcal{E}$ be the set of mappings from $\{1, \dots, d\}$ to the non-negative integers; to $\mathbf{q} \in \mathcal{E}$ we associate the following operators:

$$\partial_{\mathbf{q}} = \prod_{\alpha \in \{1,\dots,d\}} (\partial_\alpha)^{\mathbf{q}(\alpha)}, \quad \partial_{\mathbf{q}}^* = \prod_{\alpha \in \{1,\dots,d\}} (\partial_\alpha^*)^{\mathbf{q}(\alpha)}.$$

Duality is realized through the identities:

$$\mathbb{E}[(\partial_\alpha \varphi)\, \psi] = \mathbb{E}[\varphi\, (\partial_\alpha^* \psi)]; \quad \mathbb{E}[(\partial_{\mathbf{q}} \varphi)\, \psi] = \mathbb{E}[\varphi\, (\partial_{\mathbf{q}}^* \psi)],$$

and the commutation relationships between annihilation and creation operators are given by the Heisenberg rules:

$$\partial_\alpha^* \partial_\beta - \partial_\beta \partial_\alpha^* = \begin{cases} 1, & \text{if } \alpha = \beta \\ 0, & \text{if } \alpha \neq \beta. \end{cases}$$

The d-dimensional Hermite polynomials are indexed by $\mathcal{E}$, which means that to each $\mathbf{q} \in \mathcal{E}$ we associate

$$H_{\mathbf{q}}(\xi) := (\partial_{\mathbf{q}}^* 1)(\xi) = \prod_\alpha H_{\mathbf{q}(\alpha)}(\xi^\alpha).$$

Let $\mathbf{q}! = \prod_\alpha \mathbf{q}(\alpha)!$. Then

$$\left\{ H_{\mathbf{q}} / \sqrt{\mathbf{q}!} \right\}_{\mathbf{q} \in \mathcal{E}}$$

is an orthonormal basis of $L^2(\gamma_d)$. Defining operators ε_β on $\mathcal{E}$ by

$$(\varepsilon_\beta \mathbf{q})(\alpha) = \mathbf{q}(\alpha), \quad \text{if } \alpha \neq \beta;$$
$$(\varepsilon_\beta \mathbf{q})(\beta) = \begin{cases} \mathbf{q}(\beta) - 1\,, & \text{if } \mathbf{q}(\beta) > 0; \\ \quad 0 \quad , & \text{otherwise,} \end{cases}$$

we get

$$\partial_\beta H_{\mathbf{q}} = \mathbf{q}(\beta)\, H_{\varepsilon_\beta \mathbf{q}}. \tag{1.6}$$

In generalization of the one-dimensional case given in Proposition 1.1 we now have the analogous d-dimensional result.

Proposition 1.3. *A function φ with all its partial derivatives in $L^2(\gamma_d)$ has the following representation by a series converging in $L^2(\gamma_d)$:*

$$\varphi = \sum_{\mathbf{q}\in\mathcal{E}} \frac{1}{\mathbf{q}!}\,\mathbb{E}[\partial_{\mathbf{q}}\varphi]\,H_{\mathbf{q}}\ . \tag{1.7}$$

Corollary 1.4. *For $c \in \mathbb{R}^d$ denote*

$$\|c\|^2 = \sum_\alpha (c^\alpha)^2, \quad (c\,|\,\xi) = \sum_\alpha c^\alpha\,\xi^\alpha, \quad c^{\mathbf{q}} = \prod_\alpha (c^\alpha)^{\mathbf{q}(\alpha)}.$$

Then we have

$$\exp\left((c\,|\,\xi) - \frac{1}{2}\|c\|^2\right) = \sum_{\mathbf{q}\in\mathcal{E}} \frac{c^{\mathbf{q}}}{\mathbf{q}!}\,H_{\mathbf{q}}(\xi)\ . \tag{1.8}$$

In generalization of the one-dimensional case (1.3) the number operator is defined by

$$\mathcal{N} = \sum_{\alpha\in\{1,\ldots,d\}} \partial^*_\alpha\partial_\alpha, \tag{1.9}$$

thus

$$(\mathcal{N}\varphi)(\xi) = \sum_{\alpha\in\{1,\ldots,d\}} (-\partial^2_\alpha\varphi + \xi^\alpha\partial_\alpha\varphi)(\xi), \quad \xi\in\mathbb{R}^d\ . \tag{1.10}$$

In particular, we get $\mathcal{N}(H_{\mathbf{q}}) = |\mathbf{q}|\,H_{\mathbf{q}}$ where $|\mathbf{q}| = \sum_\alpha \mathbf{q}(\alpha)$.

Denote by $C^k_b(\mathbb{R}^d)$ the space of k-times continuously differentiable functions on $\mathbb{R}^d$ which are bounded together with all their first k derivatives. Fix $p \geq 1$ and define a Banach type norm on $C^k_b(\mathbb{R}^d)$ by

$$\begin{aligned}\|f\|^p_{D^p_k} := \int_{\mathbb{R}^d} \Bigg(&|f|^p + \sum_{\alpha\in\{1,\ldots,d\}} |\partial_\alpha f|^p \\ &+ \sum_{\alpha_1,\alpha_2\in\{1,\ldots,d\}} |\partial^2_{\alpha_1,\alpha_2} f|^p + \ldots + \sum_{\alpha_i\in\{1,\ldots,d\}} |\partial^k_{\alpha_1,\ldots,\alpha_k} f|^p\Bigg)\,d\gamma_d\ .\end{aligned} \tag{1.11}$$

A classical fact (see for instance [143]) is that the completion of $C^k_b(\mathbb{R}^d)$ in the norm $\|\cdot\|_{D^p_k}$ is the Banach space of functions for which all derivatives up to order k, computed in the sense of distributions, belong to $L^p(\gamma_d)$. We denote this completion by $D^p_k(\mathbb{R}^d)$.

Theorem 1.5. *For any $f \in C^2_b(\mathbb{R}^d)$ such that $\int f d\gamma_d = 0$ we have*

$$\|\mathcal{N}(f)\|_{L^2(\gamma_d)} \leq \|f\|_{D^2_2} \leq 2\,\|\mathcal{N}(f)\|_{L^2(\gamma_d)}\ . \tag{1.12}$$

Proof. We use the expansion of f in Hermite polynomials:

$$\text{if } f = \sum_{\mathbf{q}} c_{\mathbf{q}} H_{\mathbf{q}} \quad \text{then} \quad \|f\|^2_{L^2(\gamma_d)} = \sum_{\mathbf{q}} \mathbf{q}!\,|c_{\mathbf{q}}|^2\ .$$

By means of (1.9) we have

$$\|\mathcal{N}(f)\|^2_{L^2(\gamma_d)} = \sum_{\mathbf{q}} |\mathbf{q}|^2\, \mathbf{q}!\, |c_{\mathbf{q}}|^2 \,.$$

The first derivatives $\partial_\alpha f$ are computed by (1.6) and their $L^2(\gamma_d)$ norm is given by

$$\sum_\alpha \int_{\mathbb{R}^d} |\partial_\alpha f|^2 \, d\gamma_d = \sum_{\mathbf{q}} |c_{\mathbf{q}}|^2 \mathbf{q}! \sum_\alpha \mathbf{q}(\alpha) = \sum_{\mathbf{q}} |c_{\mathbf{q}}|^2 \mathbf{q}!\, |\mathbf{q}| \,.$$

The second derivatives $\partial^2_{\alpha_1,\alpha_2} f$ are computed by applying (1.6) twice and the $L^2(\gamma_d)$ norm of the second derivatives gives

$$\sum_{\alpha_1,\alpha_2} \int_{\mathbb{R}^d} |\partial^2_{\alpha_1,\alpha_2} f|^2 \, d\gamma_d = \sum_{\mathbf{q}} |c_{\mathbf{q}}|^2 \mathbf{q}! \sum_{\alpha_1,\alpha_2} \mathbf{q}(\alpha_1)\mathbf{q}(\alpha_2) = \sum_{\mathbf{q}} |c_{\mathbf{q}}|^2 \mathbf{q}!\, |\mathbf{q}|^2 \,.$$

Thus we get

$$\|f\|^2_{D^2_2} = \sum_{\mathbf{q}} |c_{\mathbf{q}}|^2 \mathbf{q}!\, (1 + |\mathbf{q}| + |\mathbf{q}|^2) \,.$$

As we supposed that $c_0 = 0$ we may assume that $|\mathbf{q}| \geq 1$. We conclude by using the inequality $x^2 < 1 + x + x^2 < 4x^2$ for $x \geq 1$. □

1.2 Wiener Space as Limit of its Dyadic Filtration

Our objective in this section is to approach the financial setting in continuous time. Strictly speaking, of course, this is a mathematical abstraction; the time series generated by the price of an asset cannot go beyond the finite amount of information in a sequence of discrete times. The advantage of continuous-time models however comes from two aspects: first it ensures stability of computations when time resolution increases, secondly models in continuous time lead to simpler and more conceptual computations than those in discrete time (simplification of Hermite expansion through iterated Itô integrals, Itô's formula, formulation of probabilistic problems in terms of PDEs).

In order to emphasize the fact that the financial reality stands in discrete time, we propose in this section a construction of the probability space underlying the Brownian motion (or the Wiener space) through a coherent sequence of discrete time approximations.

We denote by $\mathscr{W}$ the space of continuous functions $W\colon [0,1] \to \mathbb{R}$ vanishing at $t = 0$. Consider the following increasing sequence $(\mathscr{W}_s)_{s\in\mathbb{N}}$ of subspaces of $\mathscr{W}$ where $\mathscr{W}_s$ is constituted by the functions $W \in \mathscr{W}$ which are linear on each interval of the dyadic partition

$$[(k-1)2^{-s}, k2^{-s}], \quad k = 1, \ldots, 2^s \,.$$