Christopher D. Hollings · Mark V. Lawson

Wagner's Theory of Generalised Heaps

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Preface

In 1953, the Saratov-based differential geometer Viktor Vladimirovich Wagner (1908–1981) published an 88-page paper entitled 'Theory of generalised heaps and generalised groups' in which he outlined abstract theories for the mathematical objects in his title, as well as exploring their mutual connections. The development of generalised groups (or inverse semigroups, as we now usually term them) came from his efforts to axiomatise systems of injective partial mappings of a set, this in turn having been motivated by the desire to derive an abstract version of the so-called 'pseudogroups' that Veblen and Whitehead had introduced as tools in differential geometry. Inverse semigroups have subsequently gone on to be one of the most-studied classes of semigroups and have found applications in a range of branches of mathematics. Wagner's 'generalised heaps', however, remain considerably less well known, at least to non-Russian-reading mathematicians. The derivation of these too was motivated by considerations from differential geometry, this time a desire to axiomatise coordinate atlases, in which context it is no longer appropriate to consider a binary operation in the abstract setting but a *ternary* one. 'Generalised heaps', along with the related 'heaps' and 'semiheaps', are thus abstract systems equipped with ternary operations obeying certain axioms.

Having arrived at his various classes of heaps and semiheaps via considerations from differential geometry, however, Wagner realised that these ideas may in fact be cast in a rather more general setting. An important notion in Wagner's derivation of generalised *groups* was that of a binary relation and, in particular, the composition of two such: it was the realisation that the composition of partial mappings is a special case of that of binary relations that had enabled Wagner to overcome some of the difficulties encountered by previous mathematicians who had worked in this area. Wagner was thus led to the study of $\mathfrak{P}(A \times B)$: the collection of all binary relations between the elements of two sets A and B (i.e., all subsets of $A \times B$). When A and B are distinct, we may not apply the usual binary composition of relations but must instead use a natural ternary operation. Under this operation, $\mathfrak{P}(A \times B)$ forms a semiheap. Wagner investigated certain subsets of $\mathfrak{P}(A \times B)$, most particularly $\mathfrak{K}(A \times B)$, the collection of all injective partial mappings from A to B, which he showed to be a generalised heap under the considered ternary operation. Indeed,

$\mathfrak{K}(A \times B)$ serves as the concrete model for generalised heaps: any generalised heap can be embedded into one of this form.

What is particularly beautiful about Wagner's theory of generalised heaps, however, are its links with the parallel theory of generalised groups. In essence, if we consider $\mathfrak{P}(A \times B)$ and its subsets for $A \neq B$, then we must use the language of heaps, semiheaps and generalised heaps, but as soon as we set $A = B$, groups, semigroups and generalised groups appear. Wagner was able to exploit this symmetry to develop two interlinked theories that provide abstract descriptions of systems of binary relations on sets and thus of various systems of mappings between sets.

Although a certain number of Wagner's notions have filtered through into the mathematical community via the papers of others, there remain many ideas in his 1953 paper that are not well known in the West: the bulk of the elegant theory of generalised heaps in particular. The paper appeared in the major Soviet journal *Matematicheskii sbornik* but pre-dated the routine translation of that journal into English (which began in 1967). We therefore bring this text to a wider readership by presenting a full translation. We believe that Wagner's work will be of interest not only to historians of modern algebra[1] but also to present-day algebraists, whom we hope to inspire to take up those of Wagner's notions that have not received as much attention as they might have done.

A translation of the 1953 paper forms the bulk of this book, but we also include some shorter pieces by Wagner that lend context, both mathematical and historical, to the longer work. In addition to the translations of Wagner's papers, we include a brief biography of Wagner and a list of his publications—as well as their being relevant to the present book, we offer these as a resource for future scholars, since Wagner and his work do not appear to have received the same amount of attention as some of his contemporaries. We also include here two essays: one historical and the other mathematical. The first sets Wagner's theory of generalised heaps into the mathematical context of the early 1950s. The mathematical essay relates Wagner's work to present-day mathematics, in particular to semigroup theory, and suggests fruitful avenues of research that come out of Wagner's 1953 paper.

Oxford, UK — Christopher D. Hollings
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[1]These translations are intended as a companion piece to the very brief outline of Wagner's theories that appeared in: Christopher Hollings, *Mathematics Across the Iron Curtain: A History of the Algebraic Theory of Semigroups*, History of Mathematics, vol. 43, American Mathematical Society, 2014.

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[MVL] I would like to thank Christopher Hollings for asking me to contribute to this book; it will surely help in making Wagner's work better known as it richly deserves to be. My thanks also go to Jonathon Funk, Chris Heunen, Philip Scott and Benjamin Steinberg for all commenting on a first draft of Chapter 9. Special thanks go to Jean Pradines for describing some of the story behind the geometry merely touched upon there. Needless to say, any errors are solely my own.

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Notation

We provide here a guide to the notation employed by Wagner in his building of the theories of generalised heaps and generalised groups; much of this notation is standard, although some of it was Wagner's own invention. Most of this notation is defined for semiheaps but may also be easily applied to semigroups (e.g., $\prec$). We provide page references to the definitions of the notation in Wagner's texts, with preference given to those in Chapter 8. We do not record the details of Wagner's logical symbolism here; the reader should refer instead to his own table of notation, which is translated here on p. 54, and to the brief notes that precede each translation. Similarly, we do not include his set-theoretic notation here, not only because it already appears in a table on p. 55 but also because it is entirely standard.

Binary Relations, Mappings and Transformations

$\overset{-1}{\alpha}$	The inverse of a binary relation α, as defined in Chapter 4	p. 61
α^{-1}	The inverse of a function α in its conventional sense	
$\beta \circ \alpha$	The composition (product) of two binary relations $\alpha \subset K \times L$, $\beta \subset L \times M$; also applied in the special case of composition of transformations	p. 62
$\overset{n}{\alpha}$	The nth power of a binary relation α	p. 67
$\widetilde{\alpha}$	The image of a binary relation $\alpha \subset K \times L$ under a mapping $K \to L$	p. 115
$\mathrm{pr}_1\,\alpha$	The first projection of a binary relation α, as defined in Chapter 4; also applied in the special case of a mapping or transformation	p. 28
$\mathrm{pr}_2\,\alpha$	The second projection of a binary relation α, as defined in Chapter 4; also applied in the special case of a mapping or transformation	p. 28

$\alpha\langle k\rangle$	The set of all images of $k \in K$ under a binary relation $\alpha \subset K \times L$ as a multivalued function; reduces to the class of k in the case of a binary relation on a single set	p. 60
$\alpha(k)$	The same as $\alpha\langle k\rangle$ but in the case where α is single-valued	p. 60
$\lambda_{k_1 k_2}$	The right translation corresponding to a pair of elements k_1, k_2 of a semiheap	p. 90
λ_g	The right translation corresponding to an element g of a semigroup	p. 133
$\widetilde{\lambda}_g$	The reduced right translation corresponding to an element g of a generalised group	p. 135
$\mu_{k_1 k_2}$	The left translation corresponding to a pair of elements k_1, k_2 of a semiheap	p. 90
μ_g	The left translation corresponding to an element g of a semigroup	p. 133
$\widetilde{\mu}_g$	The reduced left translation corresponding to an element g of a generalised group	p. 135
$(\varphi_3\varphi_2\varphi_1)$	The triple product of binary relations φ_1, φ_2, φ_3	p. 33
$\langle\kappa_3\kappa_2\kappa_1\rangle$	The majorant triple product of coordinate systems κ_1, κ_2, κ_3	p. 35

Special Binary Relations, Mappings and Transformations

Δ_A	As a binary relation, the equality relation on a set A; as a transformation, the identity transformation on A	p. 65
ω	The canonical order relation in a generalised heap or generalised group; more often denoted by $\prec$	p. 100
ω_C	The canonical order relation in $C(K)$	p. 107
$\prec$	The canonical order relation in a generalised heap; denoted initially by ω	p. 101
ρ	The generalised invertibility relation in a semigroup	p. 81
σ	The compatibility relation in a generalised heap or generalised group	p. 92
τ	The transitivity relation between the elements of two sets	p. 122
τ_λ	The transitivity relation for the semigroup of right translations of a semigroup	p. 135
τ_μ	The transitivity relation for the semigroup of left translations of a semigroup	p. 135
$\widetilde{\tau}$	The mutual transitivity relation for a semigroup of partial transformations	p. 138

Collections of Binary Relations, Mappings and Transformations

$\mathfrak{P}(A)$	The set of all subsets of a given set A	p. 54
$\mathfrak{P}(\Delta_A)$	The set of all partial identity transformations in $\mathfrak{K}(A \times A)$	p. 121
$\mathfrak{P}(A \times B)$	The semiheap of all binary relations between sets A and B	p. 118
$\mathfrak{P}(A \times A)$	The involuted semigroup of all binary relations on a set A	p. 118
$\mathfrak{F}(A \times B)$	The set of all partial mappings from a set A to a set B	p. 119
$\mathfrak{F}(A \times A)$	The semigroup of all partial transformations of a set A	p. 119
$\mathfrak{F}_1(A \times B)$	The set of all mappings from a set A into a set B	p. 119
$\mathfrak{F}_1(A \times A)$	The semigroup of all transformations of a set A	p. 119
$\mathfrak{G}(A \times A)$	The group of all bijections of a set A	p. 40
$\mathfrak{K}(A \times B)$	The generalised heap of all one-to-one partial mappings from a set A to a set B	p. 120
$\mathfrak{K}(A \times A)$	The generalised group of all one-to-one partial transformations of a set A	p. 121
$\mathfrak{M}(A \times B)$	Used by Wagner in earlier papers for $\mathfrak{K}(A \times B)$	p. 33
$\mathfrak{M}(A \times A)$	Used by Wagner in earlier papers for $\mathfrak{K}(A \times A)$	p. 38
$\Lambda(K)$	The semigroup of all right translations of a semiheap K	p. 90
$M(K)$	The semigroup of all left translations of a semiheap K	p. 90

Semiheaps, Semigroups and Their Elements and Subsets

$\overline{g}$	The generalised inverse (not necessarily unique) of an element g of a semigroup; also used to denote a closure operator (see below) or, less often,	p. 81
	simply a generic element	p. 38
g^{-1}	The result of applying an involution $^{-1}$ to an element g of an involuted semigroup; more specifically, the unique generalised inverse of an element g of a generalised group	p. 71
$\mathfrak{g}^{-1}$	The set $\{g^{-1} : g \in \mathfrak{g}\}$ corresponding to a subset $\mathfrak{g}$ of a semigroup with involution $^{-1}$	p. 71
I	The subset of idempotents of a generalised group	p. 81
$[k_1 k_2 k_3]$	The result of the standard ternary operation in a semiheap	p. 56

$k^{[2n+1]}$	Shorthand for $[k \cdots k]$, where k appears $2n+1$ times	p. 57
$[[\, k_1 k_2 k_3 \,]]$	The result of the inverted ternary operation defined in a semiheap by (8.1.6)	p. 57
$k_1 \top k_2$	The result of the idempotent binary operation defined in a generalised heap by (8.4.15)	p. 98
$\omega(\mathfrak{k})$	The majorant of a subset $\mathfrak{k}$ of a generalised heap	p. 109
$\overset{-1}{\omega}(\mathfrak{k})$	The minorant of a subset $\mathfrak{k}$ of a generalised heap	p. 109

Closure Operators

$\overline{x}$	Generic closure operator, used in a range of senses:	p. 55
	• Intersection of all subsets containing a given subset	p. 55
	• Stable closure of a subset	p. 58
	• (Left/right/lateral) ideal closure of a subset	p. 59
	• Diagonal closure of a binary relation	p. 65
	• Involution-invariant closure of a subset of an involuted semigroup;	p. 72
	also used for the generalised inverse of an element in	p. 81
	a semigroup or, occasionally, a generic element	p. 38
$\overset{d-}{\rho}$	The diagonal closure of a binary relation ρ: the smallest diagonally semi-invariant relation containing ρ; note that this is also denoted in places by $\overline{\rho}$	p. 65
$\overset{e-}{\rho}$	The equivalence closure of a binary relation ρ: the smallest equivalence relation containing ρ	p. 67
$\mathfrak{o}(K)$	The ideal closure of $[K\{e\}K]$ in $U_e(K)$	p. 79

Special Semiheaps and Semigroups

$\mathfrak{P}(K)$	The semiheap of all subsets of a semiheap K	p. 58
$C(K)$	The canonical embedding semigroup for a semiheap K	p. 79
$C_e(K)$	The canonical embedding semigroup with identity for a semiheap K	p. 79
$C_0(K)$	The reduced canonical embedding generalised group for a generalised heap K	p. 92
$\mathfrak{C}(K)$	The bicommutative semiheap of all compatible subsets of a semiheap K	p. 93

$\mathfrak{C}(G)$	The idempotent-commutative involuted semigroup of all compatible subsets of a generalised group G	p. 95
$\mathfrak{D}(K)$	The complete generalised heap forming the image of $\mathfrak{C}(K)$ under the minorant closure operation	p. 109
K_e	The involuted semigroup obtained from a semiheap K with biunitary element e	p. 77
$L(M)$	The free semiheap generated by a set M	p. 71
$\mathfrak{M}(K)$	The semiheap of all minorantly closed subsets of a generalised heap K	p. 109
$\mathfrak{M}(G)$	The involuted semigroup of all minorantly closed subsets of a generalised group G	p. 109
$\mathfrak{R}(K)$	The generalised heap of all compatible generalised subheaps of a generalised heap K	p. 96
$\mathfrak{R}(G)$	The generalised group of all generally invertible elements of $\mathfrak{C}(K)$	p. 96
$U(K)$	The universal embedding semigroup for a semiheap K	p. 79
$U_e(K)$	The universal embedding semigroup with identity for a semiheap K	p. 78

Chapter 1
Introduction

As we have already noted in the Preface, Wagner's theory of generalised heaps[1] (and of the related heaps and semiheaps) is little-known in the West, it being a victim of the communications difficulties that assailed Cold War mathematics.[2] Having been published in major Soviet journals, Wagner's work did find its way into Western libraries, but appears for the most part to have gone unread, owing to the fact that it was in Russian. The parallel theory of generalised groups (inverse semigroups) is rather better known in the West, but this is arguably only because of its independent development at the hands of G. B. Preston (1925–2015) in the UK [12–14]. Inverse semigroups are now one of the most-studied classes of semigroups, and their theory is well known the world over, thanks to such monographs as [11] and [9]. A small number of summaries of Wagner's ideas concerning generalised heaps have attempted to bring them to a wider readership [1, 6, 15], but no comprehensive treatment or translation has hitherto been available—there has, however, been a little latter-day Western research into the concept of a generalised heap [2, 4, 5, 10], which complements the small number of works that appeared by Soviet authors in the 1950s and 1960s. In light of this evident interest, as well as the great relevance of Wagner's research to present-day mathematics (see Chapter 9), we believe that the time is ripe for a fuller account of his work, and, since Wagner was a particularly lucid writer, the best way to do this is through a direct translation of his own words. Moreover, the production of this translation is (reasonably) timely, since 2016 marked the 150th anniversary of the founding of *Matematicheskii sbornik*, the journal in which Wagner's major work on this subject appeared.

[1] On the transliteration of 'Wagner', see note 1 on p. 5; on the translation of the term 'heap', see p. 25.

[2] On these communications difficulties in the context of semigroup theory, see [7]. For a broader perspective, see [8]—Chapter 4 in particular deals with the language barrier.

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The pieces that we choose to include here are the following:

- ‘A ternary algebraic operation in the theory of coordinate structures’ [16];
- ‘On the theory of partial transformations’ [17];
- ‘Generalised groups’ [18];
- **‘Theory of generalised heaps and generalised groups’ [19].**

These appear here as Chapters 5–8, respectively. The centrepiece of the present book is the translation of Wagner’s main work on this subject, marked here in bold, which is considerably longer than the other articles, and presents the parallel theories of generalised heaps and generalised groups in all their mutual connections.

The first three items on the above list are short reports that were presented to the Soviet Academy of Sciences, ahead of the publication of Wagner’s main work. Since there was a delay in the acceptance of the last item on the above list [7, p. 262], he announced some of his results in the second and third items; these are included here partly for historical interest, since they were the first papers to introduce to the world the core ideas connected with inverse semigroups, and partly because they serve as a useful introduction to some of the mathematics of the much longer paper that follows. The first item, on the other hand, provides an indication of how these ideas link back to differential-geometric considerations, something that Wagner did not go into in the longer paper; it also has a relevance for the further discussion of these ideas in Section 9.1. The translations are preceded by a general discussion of the choices made during the process of translating them from Russian (Chapter 4). Moreover, for the convenience of the reader, each individual translation is preceded by a summary, some points of note, and an indication of how that paper links to the others; a guide to Wagner’s notation appears at the beginning of the book.

We endeavour to present Wagner’s ideas with as much context as possible. To that end, we provide a biographical sketch of Wagner (Chapter 2), as well as a list of his publications (Appendix A), including cross-references to *Mathematical Reviews*, *Zentralblatt für Mathematik und ihre Genzgebiete* and *Jahrbuch über die Fortschritte der Mathematik* wherever possible. In particular, we present these as resources for future scholars. Wagner has been somewhat overlooked, as compared with some of his contemporaries—for example, Charles Ehresmann (1905–1979), who worked on similar topics to Wagner, and whose *Oeuvres complètes* [3] was published after his death—and we hope therefore to spark further interest in Wagner and his work. Indeed, the publications list contains a great range of further work by Wagner that might yet be explored.

Additional context for the translations is provided here by two essays. In the first (Chapter 3), we consider the place of Wagner’s work on generalised heaps within twentieth-century mathematics: its links both to differential geometry and to algebra, including its connection with prior investigations of systems with ternary operations. The second essay (Chapter 9) relates Wagner’s work to present-day mathematical research, and points to avenues that might yet be explored.

For ease of reference, a uniform numbering has been imposed on top of that of Wagner’s papers. In order to recover Wagner’s original theorem or equation numbers from our numbering, the first digit must be deleted: thus, for example, our Theorem 7.1 within the translation of Wagner's paper 'Generalised groups' (Chapter 7) appears simply as 'Theorem 1' in the original. The same principle applies also to the longer 'Theory of generalised heaps and generalised groups' (Chapter 8), which, unlike the other pieces, is divided into sections, and therefore has an extra layer of numbering: our Theorem 8.1.1 on p. 56 appears as 'Theorem 1.1' in the original, and so on.

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Chapter 2
Viktor Vladimirovich Wagner (1908–1981)

Viktor Vladimirovich Wagner (Виктор Владимирович Вагнер)[1] was born in Saratov on 4 November 1908. His early training was as a teacher, a career that he pursued at the end of the 1920s and the beginning of the 1930s, during which time he also taught himself the rudiments of mathematics and physics — his social background was such that, under the Soviet system, he had no access to higher education. He was eventually permitted to sit the final examinations in physics and mathematics at Moscow State University, and was awarded a university diploma in 1930.

At this time, Wagner's interests leaned more towards physics than pure mathematics, with relativity being of particular appeal. He hoped to become a graduate student of the physicist I. E. Tamm. However, it was not possible for Tamm to take on students in this area, since relativity had been deemed a 'pseudoscience'. Wagner was advised instead to pursue a mathematical topic close to relativity, namely differential geometry, under the supervision of V. F. Kagan. In 1934, he submitted a thesis entitled *Differential geometry of non-holonomic manifolds* for the candidate degree (equivalent to a Western PhD). A Western visitor, J. A. Schouten (whom we shall meet again in Section 3.4), served as Wagner's opponent, and recommended that the thesis receive the higher Soviet doctoral degree.

Following the award of his doctorate, Wagner took up the newly established chair in geometry at Saratov State University; he remained in this position until his retirement in 1978. He died in Brest (in present-day Belarus) on 15 August 1981 whilst returning from a foreign trip.

In the earlier part of his career, Wagner's work was focused firmly on problems from differential geometry: for example, the properties of non-holonomic manifolds. A little later, he developed geometric methods for solving variational

[1]This very brief biographical sketch is adapted from that in [2, §10.3], and is based upon the resources listed on pp. 182–184 of the present book. Note that we choose to transliterate 'Вагнер' as 'Wagner' in light of the comments in [4, p. 152].

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problems, and worked at the intersection of the geometric theory of PDEs and the calculus of variations. At the end of the 1940s, Wagner was drawn into the study of the foundations of differential geometry (for reasons that we will explore in Section 3.4), and from there to the algebraisation of those foundations. The work that appears in translation in the present volume came directly from Wagner's search for algebraic constructs that would serve as abstract models for structures appearing in the differential-geometric context.

Under Wagner's influence, Saratov gained a reputation as a centre for geometry and algebra (see [1] and [3]), with seminars on these topics that attracted researchers from across the USSR. Moreover, Wagner supervised over 40 dissertations in geometry, the calculus of variations and algebra. Thus, his influence on Soviet mathematics was quite pronounced, a fact that was recognised by his being awarded the Order of Lenin, the Order of the Red Banner and the title of Honoured Scientist of the RSFSR. However, as noted in Chapter 1, his contributions to mathematics have been somewhat overlooked by mathematicians outside the former USSR, as well as by historians of twentieth-century mathematics. We hope therefore that the material included in the present volume will begin to correct this oversight.

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Chapter 3
Wagner's Work in Historical Context

As may be seen from a perusal of Chapters 5–8, Wagner's work on generalised heaps and generalised groups pulled together ideas from a range of mathematical disciplines. Sparked initially by considerations from differential geometry, much of Wagner's research was cast in the language of the nascent theory of semigroups, but it also drew upon the notions of ternary operations, partial transformations, and, most particularly, binary relations. In this chapter, we outline very brief histories of the above-mentioned concepts, and describe how they came together in Wagner's work.[1] We begin the chapter with a short reminder of the modern formulation of basic notions relating to inverse semigroups.[2]

3.1 Modern Theory

Within a given semigroup S, we may define a general notion of 'inverse' as follows: an element $s \in S$ has (*generalised*) *inverse* $s' \in S$ if

$$ss's = s \qquad \text{and} \qquad s'ss' = s'. \tag{3.1}$$

In this situation, we also say that s is an inverse for s'. Note that if S is a monoid, then we do not necessarily have that $ss' = 1 = s's$; all we can say is that ss' and $s's$ are idempotent (and, in general, distinct).

[1]This chapter was constructed with particular help from [6, 34, 43], and [56]. Sections 3.4 and 3.6 draw heavily upon [30, §§10.2, 10.4], whilst parts of Sections 3.1 and 3.5 paraphrase [30, §10.1] and [29], respectively. A good summary of Wagner's work on heaps, semiheaps and generalised heaps that goes into considerably more detail than our Section 3.6 is [2]; for a broader survey of ternary operations and semigroups, see [28].

[2]For greater detail on inverse semigroups, see [11, 32, 34, 41].

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