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Editors

Integral Methods in Science and Engineering, Volume 1

Theoretical Techniques

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Preface

The international conferences on Integral Methods in Science and Engineering (IMSE), started in 1985, are attended by researchers in all types of theoretical and applied fields, whose output is characterized by the use of a wide variety of integration techniques. Such methods are very important to practitioners as they boast, among other advantages, a high degree of efficiency, elegance, and generality.

The first 13 IMSE conferences took place in venues all over the world:

1985, 1990: University of Texas at Arlington, USA
1993: Tohoku University, Sendai, Japan
1996: University of Oulu, Finland
1998: Michigan Technological University, Houghton, MI, USA
2000: Banff, AB, Canada (organized by the University of Alberta, Edmonton)
2002: University of Saint-Étienne, France
2004: University of Central Florida, Orlando, FL, USA
2006: Niagara Falls, ON, Canada (organized by the University of Waterloo)
2008: University of Cantabria, Santander, Spain
2010: University of Brighton, UK
2012: Bento Gonçalves, Brazil (organized by the Federal University of Rio Grande do Sul)
2014: Karlsruhe Institute of Technology, Germany

The 2016 event, the fourteenth in the series, was hosted by the University of Padova, Italy, July 25–29, and gathered participants from 26 countries on five continents, enhancing the recognition of the IMSE conferences as an established international forum where scientists and engineers have the opportunity to interact in a direct exchange of promising novel ideas and cutting-edge methodologies.

The Organizing Committee of the conference was comprised of

Massimo Lanza de Cristoforis (University of Padova), chairman,
Matteo Dalla Riva (The University of Tulsa)
Mirela Kohr (Babes–Bolyai University of Cluj–Napoca),
Pier Domenico Lamberti (University of Padova),

Flavia Lanzara (La Sapienza University of Rome), and
Paolo Musolino (Aberystwyth University),

assisted by Davide Buoso, Gaspare Da Fies, Francesco Ferraresso, Paolo Luzzini, Riccardo Molinarolo, Luigi Provenzano, and Roman Pukhtaievych.

IMSE 2016 maintained the tradition of high standards set at the previous meetings in the series, which was made possible by the partial financial support received from the following:

The International Union of Pure and Applied Physics (IUPAP)
GruppoNazionale per l'Analisi Matematica, la Probabilità e le loro Applicazioni (GNAMPA), INDAM
The International Society for Analysis, Its Applications and Computation (ISAAC);
The Department of Mathematics, University of Padova

The participants and the Organizing Committee wish to thank all these agencies for their contribution to the unqualified success of the conference.

IMSE 2016 included four minisymposia:

Asymptotic Analysis: Homogenization and Thin Structures; organizer: M.E. Pérez (University of Cantabria)
Mathematical Modeling of Bridges; organizers: E. Berchio (Polytechnic University of Torino) and A. Ferrero (University of Eastern Piedmont)
Wave Phenomena; organizer: W. Dörfler (Karlsruhe Institute of Technology)
Wiener-Hopf Techniques and Their Applications; organizers: G. Mishuris (Aberystwyth University), S. Rogosin (University of Belarus), and M. Dubatovskaya (University of Belarus)

The next IMSE conference will be held at the University of Brighton, UK, in July 2018. Further details will be posted in due course on the conference web site blogs.brighton.ac.uk/imse2018.

The peer-reviewed chapters of these two volumes, arranged alphabetically by first author's name, are based on 58 papers from among those presented in Padova. The editors would like to thank the reviewers for their valuable help and the staff at Birkhäuser-New York for their courteous and professional handling of the publication process.

Tulsa, OK, USA — Christian Constanda
March 2017

The International Steering Committee of IMSE:

Christian Constanda (The University of Tulsa), *chairman*
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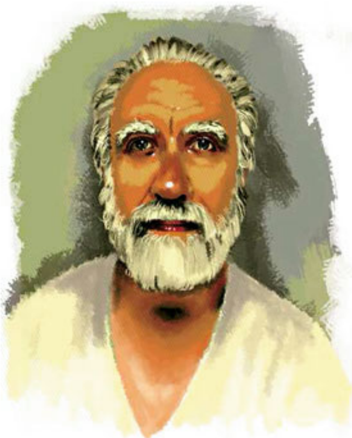
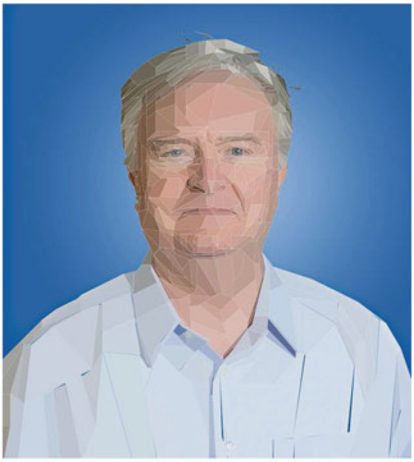
A novel feature at IMSE 2016 was an exhibition of digital art that consisted of seven portraits of participants and a special conference poster, executed by artist Walid Ben Medjedel using eight different techniques. The exhibition generated considerable interest among the participants, as it illustrated the subtle connection between digital art and mathematics. The portraits, in alphabetical order by subject, and the poster have been reduced to scale and reproduced on the next two pages.

Digital Art by Walid Ben Medjedel

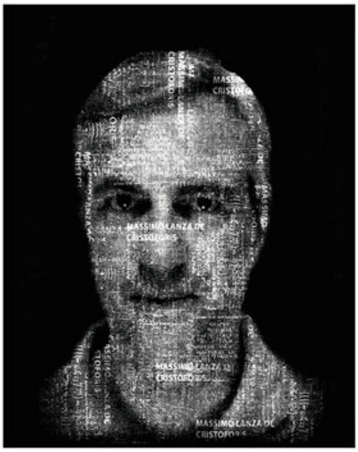


Mario Ahues
Acrilic portrait



Christian Constanda
Polygon portrait

Mirela Kohr
Vector portrait

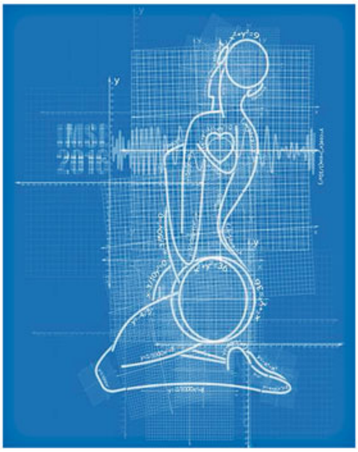


Massimo Lanza de Cristoforis
Text portrait

Flavia Lanzara
Airbrushing portrait

Dorina Mitrea
Pop-art portrait

Ovadia Shoham
Ink-pen portrait



IMSE2016 special poster
Mixed media

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Chapter 1
An L^1-Product-Integration Method in Astrophysics

M. Ahues Blanchait and H. Kaboul

1.1 Introduction

We consider a Banach space X. Let T be the integral operator defined by

$$\forall x \in X,\ \forall s \in [a,b],\ Tx(s) := \int_a^b L(s,t)H(s,t)x(t)\,dt,$$

where $(s,t) \mapsto H(s,t)$ is not smooth.

For z not in the spectrum of T, and any y in X we tackle the problem

$$\text{Find } \varphi \in X \text{ s.t. } (T - zI)\varphi = y,$$

where I denotes the identity operator on X.

The solution φ will be approximated through the exact solution φ_n of a finite rank equation

$$(T_n - zI)\varphi_n = y.$$

To do so, we propose a product-integration scheme in $X := L^1([a,b],\mathbb{C})$.

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1.2 A Product-Integration Method in $L^1([a,b],\mathbb{C})$

In the sequel, $\|x\|_1 := \displaystyle\int_a^b |x(s)|\,ds$ is the norm in $L^1([a,b],\mathbb{C})$. The subordinated operator norm is also denoted by $\|.\|_1$. The oscillation of a function x in $L^1([a,b],\mathbb{C})$, relative to a $h \in \mathbb{R}$ is

$$w_1(x,h) := \sup_{|u|\in[0,|h|]} \int_a^b |x(v+u) - x(v)|\,dv,$$

where $x(t) := 0$ for $t \notin [a,b]$. The modulus of continuity of a continuous function on $[a,b]$ is

$$w(x,h) := \sup_{u,v\in[a,b],|u-v|\le|h|} |x(u) - x(v)|.$$

The modulus of continuity of f on $[a,b]\times[a,b]$ is

$$w_2(f,h) := \sup_{u,v\in[a,b]^2,\|u-v\|\le|h|} |f(u) - f(v)|.$$

For $x \in L^1([a,b],\mathbb{C})$

$$\lim_{h\to 0} w_1(x,h) = 0.$$

T is a compact bounded linear operator from $L^1([a,b],\mathbb{C})$ into itself and $(T_n)_{n\ge 2}$ is a collectively compact approximation of T. The compactness in $L^1([a,b],\mathbb{C})$ relies on the Kolmogorov-Riesz-Fréchet theorem. We assume that

(P1) $L \in C^0([a,b]\times[a,b],\mathbb{C})$.
Let

$$c_L := \sup_{(s,t)\in[a,b]^2} |L(s,t)|.$$

(P2) H verifies:

(P2.1) $c_H := \displaystyle\sup_{t\in[a,b]} \int_a^b |H(s,t)|\,ds$ is finite.

(P2.2) $\displaystyle\lim_{h\to 0} w_H(h) = 0$,

where

$$w_H(h) := \sup_{t\in[a,b]} \int_a^b |\tilde{H}(s+h,t) - \tilde{H}(s,t)|\,ds,$$

and

$$\tilde{H}(s,t) := \begin{cases} H(s,t) & \text{for } s \in [a,b], \\ 0 & \text{for } s \notin [a,b]. \end{cases}$$

Lemma 1

$$\lim_{h \to 0+} \epsilon(H,h) = 0,$$

where

$$\epsilon(H,h) := \sup_{t \in [a,b]} \int_{b-h}^{b} |H(s,t)|\, ds.$$

Theorem 1 *Under the assumptions* **(P1)** *and* **(P2)**, *the operator* T *is linear from* $L^1([a,b],\mathbb{C})$ *into itself and compact in* $L^1([a,b],\mathbb{C})$.

Let Δ_n be a uniform grid with mesh size h_n. For $x \in L^1([a,b],\mathbb{C})$,

$$t \mapsto Q_n(x,s,t) := \frac{1}{h_n}\left[(t_{n,i}-t)L(s,t_{n,i-1}) + (t-t_{n,i-1})L(s,t_{n,i})\right]\frac{1}{h_n}\int_{t_{n,i-1}}^{t_{n,i}} x(u)\, du$$

for $i = 1,\ldots,n$, and $t \in [t_{n,i-1}, t_{n,i}]$, and $\forall x \in L^1([a,b],\mathbb{C})$, $\forall s \in [a,b]$,

$$\begin{aligned} T_n x(s) &:= \int_a^b Q_n(x,s,t)H(s,t)\, dt \\ &= \sum_{i=1}^{n} c_{n,i} w_{n,i}(s), \end{aligned}$$

where, for $i = 1,\ldots,n$,

$$c_{n,i} := \frac{1}{h_n}\int_{t_{n,i-1}}^{t_{n,i}} x(u)\, du,$$

and

$$w_{n,i}(s) := \int_{t_{n,i-1}}^{t_{n,i}} Q_n(1,s,t)H(s,t)\, dt.$$

Lemma 2 *For* $i = 1,\ldots,n$,

$$\int_a^b |w_{n,i}(s)|\, ds \le h_n c_L c_H.$$

For $h \in \mathbb{R}^+$,

$$\int_{b-h}^{b} |w_{n,i}(s)|\, ds \leq h_n c_L \epsilon(H, h),$$

$$\int_{a}^{b-h} \left|w_{n,i}(s+h) - w_{n,i}(s)\right| ds \leq h_n c_H w_2(L, h) + h_n c_L w_H(h).$$

Lemma 3 *For* $x \in L^1([a, b], \mathbb{C})$,

$$\sum_{i=1}^{n} \int_{t_{n,i-1}}^{t_{n,i}} |x(u) - c_{n,i}|\, du \leq 2w_1(x, h_n).$$

For $t \in [a, b]$,

$$\left|Q_n(1, s, t) - L(s, t)\right| \leq w_2(L, h_n).$$

Theorem 2 T_n *is a compact linear operator from* $L^1([a, b], \mathbb{C})$ *into itself and* $(T_n)_{n\geq 2}$ *is a collectively compact approximation to* T.

Lemma 4 *Let* $z \in \mathrm{re}(T)$. *For* n *large enough,* $T_n - zI$ *is invertible and it exists* $M_z > 0$ *such that*

$$\|(T_n - zI)^{-1}\|_1 \leq c_z.$$

Proof It is a consequence of the collectively compact convergence (see [An71]).

Theorem 3 *For* z *not in the spectrum of* T*, and under hypotheses* **(P1)** *and* **(P2)***, for* n *large enough,* φ_n *is uniquely defined and*

$$\|\varphi - \varphi_n\|_1 \leq c_z c_H \left(\|\varphi\|_1 w_2(L, h_n) + 2c_L w_1(\varphi, h_n)\right).$$

In the transfer equation, the kernel H is of convolution type: $a = 0$, $b = 1$ and $H(s, t) = g(|s - t|)$, where

- $\lim_{s\to 0^+} g(s) = +\infty$,
- $g \in C^0(]0, 1], \mathbb{R}) \cap L^1([0, 1], \mathbb{R})$,
- $g \geq 0$ and g is a decreasing function in $]0, 1]$.

1.3 Iterative Refinement

The approximate solution can be written as $\varphi_n = G_n(z)y$, where $G_n(z)$ is an approximate inverse of $T - zI$, and its accuracy may be improved using an iterative refinement scheme:

$$x_n^{(0)} := G_n(z)y,$$
$$x_n^{(k+1)} := x^{(0)} + (I - G_n(z)(T - zI))x_n^{(k)},$$

such as

Scheme A (Atkinson): $G_n(z) := R_n(z) := (T_n - zI)^{-1}$,
Scheme B (Brakhage): $G_n(z) := \dfrac{1}{z}(R_n(z)T - I)$,
Scheme C (Titaud): $G_n(z) := \frac{1}{z}(TR_n(z) - I)$.

Theorem 4 *If $h \to 0^+$, then*

$$\mathrm{e}(h) := 3C_H^2 C_L w(L, h) + 2C_H C_L^2 w_H(h) + 2C_H C_L^2 \epsilon(H, h) \to 0,$$

and there exist $a_z > 0$, $b_z > 0$ and $c_z > 0$ independent of n such that:
For Scheme A:

$$\frac{\|x_n^{(2\ell-1)} - \varphi\|}{\|\varphi\|} \le (b_z \mathrm{e}(h_n))^{\ell}, \quad \frac{\|x_n^{(2\ell)} - \varphi\|}{\|\varphi\|} \le a_z (b_z \mathrm{e}(h_n))^{\ell}.$$

For Scheme B:

$$\frac{\|x_n^{(k)} - \varphi\|}{\|\varphi\|} \le \big(\frac{c_z}{z}\mathrm{e}(h_n)\big)^{k+1}.$$

For Scheme C:

$$\frac{\|x_n^{(k)} - \varphi\|}{\|\varphi\|} \le \frac{c_z}{z}\big(\frac{c_z}{z}\mathrm{e}(h_n)\big)^{k}.$$

1.4 Numerical Evidence

The approximate equation is

$$\forall s \in [a, b], \quad \sum_{i=1}^{n} w_{n,j}(s) \frac{1}{h_n} \int_{t_{n,j-1}}^{t_{n,j}} \varphi_n(u)\, du - z\varphi_n(s) = y(s).$$

If we compute the local mean over $[t_{n,i-1}, t_{n,i}]$, $i = 1, \ldots, n$, we get a linear system $(A - zI)x = d$, where

$$A(i,j) := \frac{1}{h_n} \int_{t_{n,i-1}}^{t_{n,i}} w_{n,j}(s)\, ds, \ i,j = 1, \ldots, n,$$

$$d(i) := \frac{1}{h_n} \int_{t_{n,i-1}}^{t_{n,i}} y(s)\, ds,\ i = 1, \ldots, n,$$

$$x(i) := \frac{1}{h_n} \int_{t_{n,i-1}}^{t_{n,i}} \varphi_n(s)\, ds,\ i = 1, \ldots, n,$$

and

$$\varphi_n(s) = \frac{1}{z}\Big(\sum_{i=1}^{n} w_{n,j}(s) x(i) - y(s) \Big).$$

The quality of φ_n is estimated through the relative residual:

$$r(\varphi_n) := \frac{\|(T - zI)\varphi_n - y\|_1}{\|y\|_1}.$$

The radiative transfer problem describes the energy conserved by a beam radiation traveling. Let τ_* be the optical width of the medium (see [ChEtAl07]). The equation is

$$\frac{\varpi(s)}{2} \int_0^{\tau_*} E_1(|s - t|)\varphi(t)\, dt - \varphi(s) = y(s),$$

where E_1 is the first integral exponential function, the albedo is $\varpi(s) = 0.7 \exp(-s)$, and

$$y(s) = \begin{cases} -0.3 & \text{for } s \in [0, 50[, \\ 0 & \text{for } s \in [50, 100]. \end{cases}$$

The relative residual associated with φ_n has been computed both by the Kantorovich projection method and by the product-integration method. Results are shown in Table 1.1.

Since for $n \geq 100$, the computation of φ_n is too costly, we use a refinement scheme to compute a better approximate solution. The number of iterations needed to get $\|\text{Relative residual}\| \leq 10^{-12}$ is shown in Table 1.2.

Table 1.1 Relative residuals

n	Projection method	Product-integration method
10	0.0267	0.0172
20	0.0252	0.0145
50	0.0151	0.0075

Table 1.2 Number of Iterations to get $\|\text{Relative residual}\| \leq 10^{-12}$ with $n = 100$

Scheme A	Scheme B	Scheme C
124	9	9

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Chapter 2
Differential Operators and Approximation Processes Generated by Markov Operators

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2.1 Introduction

In recent years several investigations have been devoted to the study of large classes of (mainly degenerate) initial-boundary value evolution problems in connection with the possibility to obtain a constructive approximation of the associated positive C_0-semigroups by means of iterates of suitable positive linear operators which also constitute approximation processes in the underlying Banach function space. Usually, as a consequence of a careful analysis of the preservation properties of the approximating operators, such as monotonicity, convexity, Hölder continuity, and so on, it is possible to infer similar preservation properties for the relevant semigroups and, in turn, some spatial regularity properties of the solutions to the evolution problems (see, e.g., [CaEtAl99, AtCa00, Ma02, AlEtAl07, AlLe09], [AlCa94, Chapter 6] and the references therein).

More recently, by continuing along these directions, we started a research project in order to investigate the possibility of associating to a given Markov operator on the Banach space $C(K)$ of all real functions defined on a convex compact subset K of $\mathbb{R}^d$ $(d \geq 1)$ some classes of differential operators as well as some suitable positive approximation processes. The main aim is to investigate whether these differential operators are generators of positive semigroups and whether the semigroups can

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