Foundations of Engineering Mechanics

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Wave Propagation and Diffraction

Mathematical Methods and Applications

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Foreword

This monograph is devoted to topical problems of the contemporary theory of wave propagation and diffraction. It starts from the brief account of mathematical methods related to studying the problems of wave diffraction theory. Then spectral methods in the theory of wave propagation are considered in more detail. The refraction of surface gravity waves is studied with the use of the ray method that originates from geometrical optics. Various problems pertaining to the diffraction and scattering of hydrodynamic, acoustic, electromagnetic, and elastic waves by local inhomogeneities in infinite and semi-infinite domains are considered and their solutions are provided. Some aspects of the problem of generation and propagation of tsunami waves are analysed. Finally, the evolution of wave trains in a two-layer fluid is studied by the method of multiple scales.

We hope that this work would be of interest for the researchers working in the field of applied and engineering mechanics, mathematical and computational physics, hydrodynamics, etc. It would also be useful for lecturers and postgraduate students of relevant specialties.

Preface

Wave phenomena represent a fairly intriguing area of the contemporary applied mathematics and physics. Wave processes have numerous applications in hydrodynamics, electromagnetism, magnetohydrodynamics, biophysics and biomechanics, acoustics, etc. This monograph is devoted to two distinct aspects of wave dynamics: wave propagation and diffraction, with the main focus put on the wave diffraction.

Wave interaction with rough bottom surfaces (topography), offshore drilling platforms, and wave energy collectors is accompanied by the diffraction of waves. The diffraction theory lies at the interface between physics and applied mathematics. In the broad sense, wave diffraction means any deviation of the wave motion from the laws of geometrical optics. From a mathematical point of view, the purpose of the diffraction theory is to develop analytical and numerical methods for solving diffraction problems, to classify the corresponding solutions, and to investigate their properties.

It is possible to distinguish three stages in the development of the diffraction theory: (i) Fresnel (1818) formulated the Huygens–Fresnel principle that combines the geometric approach (Huygens 1690) and interference approach (Young 1800); (ii) Helmholtz (1859) gave a strict formulation of the Huygens principle and demonstrated that it results in an integral formula (as in the potential theory) that makes it possible to calculate the value of the field at some point in terms of the field values (including the field's normal derivative) on some auxiliary closed surface enclosing that point; (iii) Poincaré (1892) and Sommerfeld (1896) showed the diffraction problems to be ordinary boundary-value problems of mathematical physics. Sommerfeld (1912) also formulated the radiation conditions. Then Meixner (1948) established the boundary conditions on the edge.

The rigorous diffraction theory distinguishes three approaches: the method of surface currents, where the diffracted field is represented as a superposition of secondary spherical waves emitted by each element (the Huygens–Fresnel principle); Fourier method; method of separation of variables and Wiener–Hopf transformation. In this monograph, we apply various mathematical methods to the solution of typical problems in the theory of wave propagation and diffraction and analyse the results obtained.

Chapter 1 presents some of the methods that are useful for solving the problems of wave diffraction theory: method of separation of variables, method of power series, method of spline functions, and method of an auxiliary boundary. We also consider some algorithms for the numerical inversion of the Laplace transform, which is often used to solve the wave diffraction problems. Finally, we give a brief account of the method of multiple scales that is often used to study the propagation of transient waves.

Chapter 2 deals with spectral methods in the theory of wave propagation. The main focus is given to the Fourier methods in application to studying the Stokes (gravity) waves on the surface of an inviscid fluid. A spectral method for calculating the limiting Stokes wave with a corner at the crest is considered as well. We also briefly consider the evolution of narrow-band wave trains on the surface of an ideal finite-depth fluid. Finally, a two-parameter method for describing the nonlinear evolution of narrow-band wave trains is described by the example of the Klein–Gordon equation with a cubic nonlinearity. The problem is reduced to a high-order nonlinear Schrödinger equation for the complex amplitude of wave envelope. This equation is integrated numerically using a split-step Fourier technique to describe the evolution of quasi-solitons.

Chapter 3 presents some results of modelling the refraction of surface gravity waves in terms of the ray method that originates from geometrical optics. Wave refraction essentially depends on the bed topography. As a result, the convergence of wave energy (or its divergence) can be observed in some water areas. The ray method makes possible the discrete or analytical assignment of the bed topography. This approach can be realised in the form of a computational programme and allows the distribution of wave fronts, rays, and heights to be constructed and analysed for the case of the transition of regular waves from deep to shallow waters. The model is verified by comparing the results with exact analytical solutions and field observations. The asymptotic analysis of the nonlinear refraction theory extends the limits of applicability of the traditional theory and provides the prediction of ray bending at approaching the wave breaking conditions. Moreover, we study the anomalous refraction in caustics.

Chapter 4 is devoted to the diffraction of surface gravity waves. Main aspects of the wave diffraction theory are described. Specific aspects and methods used to solve the problems of the wave diffraction theory are described in brief. Wave diffraction by a partially submerged elliptical cylinder with elliptical front surface and by a circular submerged cylinder is considered. Ellipticity is demonstrated to have strong effect on the wave load and its extremums, depending on the wave number. Solutions to the problem of wave diffraction by a system of vertical cylinders are presented and analysed. In this case, the wave force does not attain its maximum on the front vertical cylinder because of the significant reconstruction of the diffracted wave fields in multiply connected regions. An exact solution to the problem of wave diffraction by an asymmetrically nonuniform cylindrical scatterer is derived in the case when the scatterer parameters depend on the two coordinates—

radial and angular. The scatterer inhomogeneity is demonstrated to affect the cross scattering. The method of auxiliary boundary is used to study the diffraction of waves by a vertical column of arbitrary revolution shape. The extremums of the wave force and overturning moment applied to a cone column are found as functions of the wave number. The numerical method of spline collocation is used to study the problem of diffraction of acoustic waves by an arbitrary body of revolution. The accuracy of the numerical solution is analysed. The problem of wave scattering by a truncated cone with smooth spherical ends is considered. The effect of wave incidence angle is studied.

Chapter 5 deals with the approach that is based on the repeated use of the method of images to solve the problems of stationary acoustic, electromagnetic, and elastic wave scattering and diffraction by cylindrical and spherical obstacles in a semi-infinite domain. The solution is written in terms of an infinite series of multiply diffracted fields. Explicit approximate asymptotic solutions are found and investigated for the case of distant scattered fields in the longwave approximation. The known solutions for point obstacles are obtained as special cases described by the first terms of the series.

Chapter 6 deals with some aspects of the initial-boundary-value problems of the initiation, generation, and propagation of tsunami waves. The generation of tsunami waves by bottom movements is considered. We formulate an appropriate initial-boundary-value problem and analyse the effect of the sharpness of vertical axisymmetric bottom disturbance and the disturbance duration on the generation of tsunami waves. The propagation of nonlinear waves on water and their evolution over a nonrigid elastic bottom are investigated. Some aspects and indeterminacy of the formulation of the initial-boundary-value problems dealing with the initiation and generation of tsunami waves are considered. We consider some typical types of tsunami waves that demonstrate the indeterminacy of their initiation in time because of the indeterminacy in the physical trigger mechanism of underwater earthquakes. Based on the three-dimensional formulation, evolution equations describing the propagation of nonlinear dispersive surface waves on water over a spatially inhomogeneous bottom are obtained with allowance for the bottom disturbances in time. We use the Laplace transform with respect to the time coordinate and the power series method with respect to the spatial coordinate to find a solution to the nonstationary problem of the diffraction of surface gravity waves by a radial bottom inhomogeneity that deviates from its initial position. The propagation and stability of nonlinear waves in a two-layer fluid with allowance for surface tension are analysed by the asymptotic method of multiscale expansions.

Some insights on the directions of further development of the wave diffraction theory are outlined in the conclusion.

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We regret to note recently departed Prof. Yuriy G. Kryvonos, who constantly and actively encouraged researchers in the field of wave propagation and diffraction.

Contents

Chapter 1
Some Analytical and Numerical Methods in the Theory of Wave Propagation and Diffraction

1.1 Method of Separation of Variables

The wave propagation, diffraction, and scattering in continuous media are described by a system of partial differential equations with relevant initial and matching conditions [37]. In the case of inhomogeneous media, these equations have variable coefficients that depend on spatial coordinates. In many cases, such equations can be reduced to one higher order equation [12]. Following Ref. [18], consider a linear equation

$$Lu(\boldsymbol{x},\, t) = 0. \tag{1.1}$$

Here, $\boldsymbol{x} = (x_1,\, x_2,\, x_3)$ is a vector of spatial coordinates, $t \equiv x_4$ is the temporal coordinate, L is a differential operator of order m defined in some domain D of the three-dimensional Euclidean space $\mathbb{R}_3$ and in the range $t \in [t_0,\, T]$:

$$L = \sum_{j=0}^{m} \sum_{i_1, i_2 \ldots i_j = 1} a_{i_1,\, \ldots,\, i_j}(\boldsymbol{x}) \frac{\partial^j}{\partial x_{i_1} \, \ldots \, \partial x_{i_j}}. \tag{1.2}$$

The coefficients $a_{i_1,\, \ldots,\, i_j}(\boldsymbol{x})$ are assumed to be defined for $(x_1,\, x_2,\, x_3) \in D$, symmetric with respect to indices $i_1,\, i_2,\, \ldots,\, i_j$, and differentiable in $\mathbb{R}_3$ as many times as needed. For $j = 0$ the operator L in (1.2) is defined as the operator of multiplication by u.

In what follows, we limit our consideration to a class of such equations of type (1.1) that admit the separation of variables. This means that the solution can be looked for as follows

$$u(x_1,\, x_2,\, x_3,\, t) = X_1(x_1) X_2(x_2) X_3(x_3) T(t). \tag{1.3}$$

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Substituting (1.3) into (1.1), we obtain independent ordinary differential equations for each $X_k(x_k)$, $k = 1, 2, 3, 4$:

$$\sum_{j=0}^{m} b_{kj}(x_k)\frac{\mathrm{d}^j}{\mathrm{d}x_k^j}X_k(x_k) = 0. \tag{1.4}$$

Going over from Eq. (1.1) to (1.3) and (1.4) is possible only under significant restrictions imposed on the differential operator L and the coefficients $a_{i_1,\ldots,i_j}(\boldsymbol{x})$. Suppose that the variables can also be separated in the initial and matching conditions.

The authors of Ref. [65] showed how a linear partial differential equation with variable coefficients can be reduced to an equation with constant coefficients. Consider the following second-order partial differential equation:

$$\alpha(x)\frac{\partial^2 g(x,t)}{\partial x^2} + \beta(x)\frac{\partial g(x,t)}{\partial x} + \gamma(x)g(x,t) = a\frac{\partial g(x,t)}{\partial t} + b\frac{\partial^2 g(x,t)}{\partial t^2}. \tag{1.5}$$

It can be reduced to the canonical form

$$C(X)\frac{\partial}{\partial X}\left(\frac{1}{C(X)}\frac{\partial f}{\partial X}\right) = a\frac{\partial f}{\partial t} + b\frac{\partial^2 f}{\partial t^2} \tag{1.6}$$

by simple transformations:

$$X = \int^x |\alpha(\tau)|^{-1/2}\mathrm{d}\tau \text{ and } f(X,t) = \frac{g(x,t)}{g_0(x)}, \tag{1.7}$$

provided that $g = g_0(x)$ is a nonzero equilibrium solution to Eq. (1.5) (when $a = b = 0$). The function $C(X)$ in (1.6) is completely determined by the functions $\alpha(x)$, $\beta(x)$, and $\gamma(x)$ in (1.5).

Equation (1.5) is rather general and includes the wave equation, heat conduction equation, Laplace equation, Schrödinger equation, and Fokker–Planck equation as particular cases. It was shown that for certain functions $C(X)$ any solution to Eq. (1.6) can be expressed in terms of solutions to the equation with constant coefficients,

$$\frac{\partial^2 F(X,t)}{\partial X^2} = a\frac{\partial F(X,t)}{\partial t} + b\frac{\partial^2 F(X,t)}{\partial t^2}, \tag{1.8}$$

namely,

$$f(X,t) = \sum_{n=0}^{N} f_n(X)\frac{\partial^n F(X,t)}{\partial X^n}, \tag{1.9}$$

where the functions $f_n(X)\,(n = 0, 1, 2, \ldots, N)$ and $C(X)$ satisfy a system of coupled nonlinear ordinary differential equations with constant $f_0 = f_0(X)$. The choice of N in (1.9) is rather broad, so that any given function $C(X)$ in (1.6) could be approximated by a function that satisfies the above system.

Note that solutions to ordinary differential equations of type (1.8) with variable coefficients cannot be expressed in terms of elementary or special functions in the general case, in particular in reference to distributed inhomogeneities that are of great practical interest. Below we consider some methods that allow solutions to such equations to be constructed.

1.2 Method of Power Series

Consider a localised inhomogeneity that occupies a closed domain $\Omega \subset \mathbb{R}^3$ in the $\mathbb{R}^3$ space. In this case, we deal with a typical problem of wave diffraction by a scatterer with variable properties. The difficulties of solving such a problem are related to the consideration of partial differential equations with variable coefficients even in the case of linear problems. The methods used to solve such problems include the method of exact analytical solutions that are possible in some cases [55]; the asymptotic method of Born approximations [7]; the Bremmer series method, in which the first term represents the WKB approximation [9]; and the method of power series [56].

The Born approximations (or approximations of weak scattering) can be used in the case when $(kR\,\delta p/p) \ll 1$, where the ratio $\delta p/p$ characterises the magnitude of the inhomogeneity and $kR = 2\pi R/\lambda$ is the ratio of the inhomogeneity size to the typical wavelength λ. This approach was particularly developed in Refs. [49, 73]. Two main approximations are usually considered in this regard. The Rayleigh approximation refers to the case when the inhomogeneity magnitude is not small, i.e. $\delta p/p = O(1)$, but the inhomogeneity size is small, $kR \ll 1$. The optical Rayleigh–Hans approximation holds true in the case of weak inhomogeneity whose size cannot be regarded small, $\delta p/p \ll 1, kR = O(1)$.

The Bremmer series method is discussed in Refs. [24, 68]. In this approach, the continuous functions characterising the inhomogeneity are approximated by piecewise constant functions, and the solutions for the reflected and transmitted waves are written for each particular layer. Then the passage to the limit for an infinite number of layers is performed with the separation of the first WKB term.

The method of generalised power series is most useful in analysing the structure of the scattered wave field. Under certain mild constraints on the inhomogeneity magnitude and size, solutions to the wave scattering problem can be obtained in the form of convergent power series. Thus, the method of power series can be used in the case of arbitrary inhomogeneities, in contrast to the Born approximations.

Consider a linear second-order differential equation with variable coefficients for some function $y(x)$:

$$y'' + f(x)y' + g(x)y = 0. \tag{1.10}$$

Suppose that $f(x)$ and $g(x)$ can be represented as power series in terms of integer positive powers of x, so that Eq. (1.10) could be written as

$$y'' + \left(\sum_{n=0}^{\infty} a_n x^n\right) y' + \left(\sum_{n=0}^{\infty} b_n x^n\right) y = 0. \tag{1.11}$$

We look for a solution to Eq. (1.11) in the form of power series with unknown coefficients:

$$y = \sum_{n=0}^{\infty} \alpha_n x^n. \tag{1.12}$$

Substituting (1.12) into Eq. (1.11), we get

$$\sum_{n=2}^{\infty} n(n-1)\alpha_n x^{n-2} + \sum_{n=0}^{\infty} a_n x^n \sum_{n=1}^{\infty} n\alpha_n x^{n-1} + \sum_{n=0}^{\infty} b_n x^n \sum_{n=0}^{\infty} \alpha_n x^n = 0.$$

By equating the equations at the like powers of x in the lefthand side of the above equations to zero, we come to an infinite system of recurrence equations [57]:

$$\begin{array}{ll}
x^0 & 2\cdot 1\alpha_2 + a_0\alpha_1 + b_0\alpha_0 = 0, \\
x^1 & 3\cdot 2\alpha_3 + 2a_0\alpha_2 + a_1\alpha_1 + b_0\alpha_1 + b_1\alpha_0 = 0, \\
x^2 & 4\cdot 3\alpha_4 + 3a_0\alpha_3 + 2a_1\alpha_2 + a_2\alpha_1 + b_0\alpha_2 + b_1\alpha_1 + b_2\alpha_0 = 0, \\
\vdots & \qquad \vdots \\
x^n & (n+2)(n+1)\alpha_{n+2} + Q_n(\alpha_0, \alpha_1, \ldots, \alpha_{n+1}) = 0, \\
\vdots & \qquad \vdots
\end{array} \tag{1.13}$$

Here, $Q_n(\alpha_0, \alpha_1, \ldots, \alpha_{n+1})$ is a homogeneous first-degree polynomial in variables $\alpha_0, \alpha_1, \ldots, \alpha_{n+1}$ (α_0 and α_1 being arbitrary constants).

The recurrence equations of system (1.13) can be used to determine consecutively all the coefficients $\alpha_2, \alpha_3, \ldots, \alpha_n, \ldots$, but the questions dealing with the convergence of the obtained series and the existence of solution remain to be open. The following theorem holds true [57]:

Theorem *If the series*

$$f(x) = \sum_{n=0}^{\infty} a_n x^n, \; g(x) = \sum_{n=0}^{\infty} b_n x^n$$

are convergent for $|x| < R$, *then the power series constructed in the way described above is convergent for the same* x *and is a solution to Eq. (1.11).*

In particular, if $f(x)$ and $g(x)$ are the polynomials in x, then the obtained power series is convergent at any x.

Consider a linear second-order differential equation,

$$P_0(x)y'' + P_1(x)y' + P_2(x)y = 0, \tag{1.14}$$

that can be reduced to Eq. (1.10) after the division by $P_0(x)$ (when $P_0(x) \neq 0$ in some interval $[0, h]$) with

$$f(x) = \frac{P_1(x)}{P_0(x)},\ g(x) = \frac{P_2(x)}{P_0(x)}.$$

Then a solution to Eq. (1.14) can also be sought in the power series form. In this case, it is easier to substitute expression (1.12) directly into Eq. (1.14) and equate the unknown coefficients at the same powers of x to zero without reducing Eq. (1.14) to form (1.10).

When the coefficients in the equations of form (1.14) have poles, the use of power series approach is regulated by the Fuchs theorem [63]:

Theorem *If differential equation (1.10) is such that* $f(x)$ *and* $g(x)$ *have poles at* $x = x_0$*, then its solution can be found in the form of convergent generalised power series*

$$y(x) = (x - x_0)^{\nu} \sum_{n=0}^{\infty} \alpha_n (x - x_0)^n,$$

provided that the products $(x - x_0)f(x)$ *and* $(x - x_0)g(x)$ *remain to be finite at* $x = x_0$.

In this case, the pole x_0 can be translated to the point $x = 0$ by the change of variable and the following equation satisfying the Fuchs theorem can be obtained:

$$x^2y'' + f(x)xy' + g(x)y = 0,\ y = y(x). \tag{1.15}$$

This equation can be solved by the method of generalised Frobenius power series [63] when $f(x)$ and $g(x)$ are holomorphic functions at $x < R$ (R being the convergence radius) and $x = 0$ is a regular singular point. In the latter case, the functions $f(x)$ and $g(x)$ can be expressed in the power series form in terms of integer positive powers of x or be given by polynomials. Then Eq. (1.15) can be written as

$$x^2y'' + \left(\sum_{n=0}^{\infty} a_n x^n\right)xy' + \left(\sum_{n=0}^{\infty} b_n x^n\right)y = 0, \tag{1.16}$$

and the solution to Eq. (1.16) is written in the form of convergent power series:

$$y = x^{\nu} \sum_{n=0}^{\infty} \alpha_n x^n,\ \alpha_0 = 1. \tag{1.17}$$

Hence, the problem reduces to the determination of parameter ν and coefficients α_ν from the original equation and the recurrence relations for α_ν. The corresponding fundamental system of solutions is

$$w_1 = x^{\nu_1}\varphi_1, \quad w_2 = x^{\nu_2}\varphi_2,$$

where the first coefficient α_0 can be regarded to be nonzero in view of the indeterminacy of the exponent ν of the multiplier x^ν.

Substituting series (1.17) into Eq. (1.16) and equating the coefficients at each power of x to zero, we get an infinite system of coupled algebraic equations for the exponent ν and coefficients α_n:

$$\begin{array}{ll}
x^{\nu-2} & \alpha_0 D(\nu) = \alpha_0 + [\nu(\nu-1) + a_0\nu + b_0] = 0, \\
x^{\nu-1} & \alpha_1 D(\nu+1) + \alpha_0(\nu a_1 + b_1) = 0, \\
x^{\nu} & \alpha_2 D(\nu+2) + \alpha_1[(\nu+1)a_1 + b_1] + \alpha_0(\nu a_2 + b_2) = 0, \\
\vdots & \vdots \\
x^{\nu+n-2} & \alpha_n D(\nu+n) + \alpha_{n-1}[(\nu+n-1)a_1 + b_1] + \cdots + \alpha_0(\nu a_n + b_n) = 0, \\
\vdots & \vdots
\end{array} \tag{1.18}$$

The first equation of this system is called a governing equation. Let ν_1 and ν_2 be its roots. Suppose that they are different and their difference is a noninteger number. In this case, two sets of the coefficients α_n corresponding to each root of the governing equation can be found consecutively. Hence, we get two generalised power series of form (1.17) that represent linearly independent solutions to Eq. (1.16). These solutions converge inside of a circle reaching the nearest special point of the equation apart from the point $x = 0$. The coefficient α_0, which is present as a multiplier in all the terms of the series, remains to be arbitrary, i.e. each solution is determined to within a constant multiplier. The general solution to Eq. (1.16) is written as the linear combination of these two solutions.

When the governing equation has a double root, there exists only one solution of type (1.17). The second linearly independent solution can be determined as follows. Let $y_1(x)$ be a solution to Eq. (1.10). Then the Liouville formula [63]

$$y_2(x) = y_1(x)\left(B + A\int \frac{\mathrm{d}x}{y_1^2(x)e^{\int f(x)\mathrm{d}x}}\right)$$

implies that there exists a solution that is linearly independent of the solution of type (1.17)

$$y_1 \int \frac{e^{-\int f(x)\mathrm{d}x}}{y_1^2}\mathrm{d}x. \tag{1.19}$$

Finally, consider the case when the roots of the governing equation differ from one another by an integer, i.e. $\nu_2 = \nu_1 + n$ (n being a positive integer). The coefficients of the series corresponding to the root ν_2 can be all calculated, since the polynomials $D(\nu_2 + 1), D(\nu_2 + 2), \ldots$ take nonzero values. The coefficients of the series corresponding to the root ν_1 can be calculated only up to order $(n - 1)$ because the coefficient at α_n in the equation

$$\alpha_n D(\nu_1 + n) + \alpha_{n-1}[a_1(n + \nu_1 - 1) + b_1] + \cdots + \alpha_0(a_n \nu_1 + b_n) = 0$$

for α_n is equal to zero. This equation can be reduced to the equality

$$\alpha_{n-1}[a_1(n + \nu_1 - 1) + b_1] + \cdots + \alpha_0(a_n \nu_1 + b_n) = 0. \tag{1.20}$$

If this equality holds, the coefficients $\alpha_{n+1}, \alpha_{n+2}, \ldots$ can be expressed in terms of the coefficient α_n, which remains to be undetermined. This implies that the solution corresponding to ν_1 contains two arbitrary parameters: the coefficient α_0 that is a common factor in the solution and the coefficient α_n that determines the higher coefficients $(\alpha_{n+1}, \alpha_{n+2}, \ldots)$ of the series. If equality (1.20) does not hold, the second solution can be found using formula (1.19), namely,

$$y_2 = y_1 \ln x + x^{\nu_2} \sum_{n=0}^{\infty} \beta_n x^n. \tag{1.21}$$

The coefficients β_n can be found by substituting expression (1.21) into Eq. (1.16) and equating the coefficients at the like powers of x.

In particular, the method of power series was used to solve the problems of wave diffraction by radially inhomogeneous obstacles [56].

1.3 Method of Spline Functions

Solving the problems of wave diffraction by obstacles of complex shape is related to the approximation of functions. One of the most efficient approaches in this regard is the surface interpolation with piecewise polynomials, which is used instead of constructing the high-order interpolation polynomials. Third-order (cubic) polynomial interpolation splines $S_3(x)$ are most widely used in practical applications.

Spline approximation methods are closely related to solving partial differential equations numerically by the finite-difference method with the use of the Ritz method with specially selected basis functions. An account of using this method to the investigation of wave diffraction is given in Sect. 4.7 and Ref. [36]. Here, we briefly outline the basic concepts of spline function theory [75].

Consider some partition $\Delta\colon a = x_0 < x_1 < \cdots < x_k = b$ of the interval $[a, b]$. Let $C^k = C^k[a, b]$ be a set of k times continuously differentiable functions on $[a, b]$

(k being an integer), and $C^{-1}(a, b)$ be a set of piecewise-continuous functions with points of discontinuity of the first kind.

The function $S_{n,\nu}(x)$ is called an n-order spline of defect ν (ν is an integer such that $0 \le \nu \le n+1$) with nodes on the mesh Δ if

(i) the function $S_{n,\nu}(x)$ is a polynomial of order n on each interval $[x_i, x_{i+1}]$, i.e.

$$S_{n,\nu}(x) = \sum_{\alpha=0}^{n} a_\alpha^i (x - x_i)^\alpha \text{ for } x \in [x_i, x_{i+1}], \quad i = 0, \ldots, N-1; \tag{1.22}$$

(ii) $S_{n,\nu}(x) \in C^{n-\nu}[a, b]$.

The spline definition is valid on the whole real axis if we set $a = -\infty, \quad b = +\infty$. Besides formula (1.22), the following representation is possible for the spline on each interval $[x_i, x_{i+1}]$:

$$S_{n,\nu}(x) = \sum_{\alpha=0}^{n} b_\alpha^i (x - x_{i+1})^\alpha,\ i = 0, \ldots, N-1.$$

For cubic splines of class C^2, which have been used most frequently, we can introduce an interpolation spline function that is continuous together with its derivative on each of the intervals $[x_i, x_{i+1}]$.

Consider the spline collocation method by the example of an ordinary differential equation that can be obtained after the separation of variables in the wave diffraction problems described by partial differential equations [36]. We look for a solution to the equation [75]

$$L[y(x)] \equiv y''(x) + p(x)y'(x) + q(x)y(x) = r(x),\ x \in [a, b], \tag{1.23}$$

that satisfies the following boundary conditions:

$$\alpha_1 y(a) + \beta_1 y'(a) = \gamma_1,\ \alpha_2 y(b) + \beta_2 y'(b) = \gamma_2. \tag{1.24}$$

We assume that the two-point boundary problem (1.23), (1.24) has a unique solution $y(x)$. The requirements in regard to the smoothness of $y(x)$ and constraints imposed on the given coefficients $p(x)$, $q(x)$, $r(x)$; α_i, β_i, γ_i, $i = 1, 2$ should be stipulated in each particular case.

We introduce the partition $\Delta{:}\ a = x_0 < x_1 < \cdots < x_N = b$ on $[a, b]$ and look for an approximate solution to problem (1.22), (1.23) in the form of a cubic spline $S(x)$ of class C^2 with nodes on the mesh Δ. The spline $S(x)$ is required to satisfy Eq. (1.23) in the points $\xi_k \in [a, b]$, $k = 0, \ldots, N$ (collocation conditions) and boundary conditions (1.24):

$$L[S(\xi_k)] \equiv S''(\xi_k) + p(\xi_k)S'(\xi_k) + q(\xi_k)S(\xi_k) = r(\xi_k),\ k = 0, \ldots, N, \tag{1.25}$$

$$\alpha_1 S(a) + \beta_1 S'(a) = \gamma_1, \quad \alpha_2 S(b) + \beta_2 S'(b) = \gamma_2. \tag{1.26}$$

Relations (1.25), (1.26) represent the set of algebraic equations for the spline parameters. The points ξ_k are called the collocations nodes, and their number is determined by the dimension of the space formed by splines of class C^2. This dimension is equal to $N + 3$. Since $S(x)$ satisfies two boundary conditions (1.26), the number of collocation nodes should be equal to $N + 1$. Note that the positions of collocation nodes on the interval $[a, b]$ cannot be arbitrary.

1.4 Method of an Auxiliary Boundary

The idea behind this method is that a scatterer of an arbitrary convex shape is enclosed by an auxiliary canonical surface—cylindrical in the two-dimensional case or spherical in the three-dimensional case [47]. Then the external problem is solved exactly, and the problem in the interior domain between the scatterer and the canonical surface is solved numerically. The corresponding matching conditions need to be satisfied on the scatterer's boundary.

Suppose that we need to determine the field $\boldsymbol{u}(\boldsymbol{x}, t)$ scattered by a local inhomogeneity of an arbitrary (noncanonical) shape in the infinite exterior domain (Fig. 1.1). Let us enclose the inhomogeneity by an auxiliary surface B and employ a numerical method in the domain Ω between B and the surface S of the inhomogeneity. For the problem in Ω to be correctly defined, $\boldsymbol{u}$ has to satisfy the boundary conditions on the surface B that needs to be nonreflective. Such formulations, which involve the nonreflective boundary conditions on the auxiliary boundary, were considered, in particular, for elastic and electromagnetic waves in Refs. [26, 29–32] and for acoustic waves in Ref. [20]. The method of an auxiliary boundary was also used in the problem of scattering of electromagnetic waves by an obstacle of arbitrary shape [33] and in the problem of diffraction of surface gravity waves [47]. The diffraction of surface gravity waves by inhomogeneities was studied in Ref.

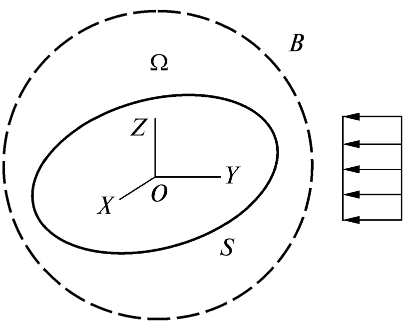

Fig. 1.1 An obstacle enclosed by an auxiliary boundary B (cylindrical or spherical)

[64] in the framework of an approximate model involving a canonical auxiliary surface. The auxiliary boundary approach was also used in studying the scattering of flexural waves by cavities in Ref. [44].

Noteworthy also are two other numerical methods that were developed in the theory of wave diffraction by obstacles of complex shape. The first one is based on the integral equation approach and was mainly used in the problems with a single obstacle [6]. The second method involves the so-called *T*-matrices (scattering matrices) [8]. The *T*-matrix method was proposed in Ref. [70]. In this approach, the field scattered by an inhomogeneity is sought in the form of expansion in terms of cylindrical (in 2D) or spherical (in 3D) functions. The matching conditions for the incident and scattered fields on the boundary result in infinite sets of equations that are represented in the form of *T*-matrices [60]. The *T*-matrix allows both the scattered and internal fields to be defined at an arbitrary point for a scatterer of arbitrary convex shape, size, and properties [71]. However, the more the scatterer shape deviates from the canonical one (cylindrical or spherical), the more terms in the expansions must be retained in order to obtain a solution with desired accuracy.

1.5 Some Algorithms for the Numerical Inversion of the Laplace Transform

The numerical inversion of the Laplace transform has been thoroughly studied, in particular, in Refs. [10, 13, 17, 19, 25, 45, 66, 74]. Some asymptotical methods were also discussed in Refs. [4, 5, 58]. The numerical inversion with the use of Laguerre polynomials was considered in Refs. [1, 22, 23, 72], and Ref. [39] deals with Jacobi polynomials. Refs. [3, 11, 14, 16, 21, 27, 28, 40, 41, 61, 67, 69] deal with the implementation of the numerical inversion of the Laplace transform in the problems of mechanics and physics.

Here, we analyse several algorithms for the numerical inversion of the Laplace transform and compare the results with reference originals and some exact solutions. It is shown that there is an optimal number of the expansion terms, i.e. the number of terms in the convergent series that can be taken into account is limited by the number of significant digits available in a particular numerical realisation. We also demonstrate that the accuracy of all the algorithms decreases for larger times. These two conclusions are the consequence of the fact that the inversion of the Laplace transform is an ill-posed problem [35, 38, 50].

The Laplace transform of function $f(t)$ is defined by the following operator for a complex parameter $p = \sigma + \mathrm{i}\tau$ [27]:

$$F(p) = \int_0^\infty f(t)e^{-pt}\mathrm{d}t, \tag{1.27}$$

provided that $F(p)$ is a uniformly convergent analytical function in the domain $\mathrm{Re}\, p > \sigma_c$ and $F(p) \to 0$ with respect to $\mathrm{Arg}\, p$ at $p \to \infty$. If (1.27) is absolutely convergent for all $\mathrm{Re}\, p > \sigma_c$, then there exists the operator of the inverse transform,

$$f(t) = \frac{1}{2\pi \mathrm{i}} \int\limits_{\sigma_c - \mathrm{i}\infty}^{\sigma_c + \mathrm{i}\infty} F(p) e^{pt} \mathrm{d}p. \tag{1.28}$$

The inverse problem consists in finding a solution $f(t)$ to the integral equation of the first kind (1.27), with $F(p)$ being a given function of complex variable p. Since the kernel $\exp(-pt)$ is a smooth function of t and p, averaging of f with weight function $\exp(-pt)$ can considerably smooth out the singularities of $f(t)$. Therefore, to recover all the local irregularities of $f(t)$, one needs to use methods that are sensitive to minor peculiarities of $F(p)$.

Since $f(t)$ is unstable with respect to small variations of $F(p)$, the inversion problem is ill-posed: its solutions do not exist for some values of numerical or functional parameters, and small variations of these parameters can lead to large variations of the solution [59, 62]. This is the main reason why all the known inversion algorithms are limited. In order to keep the inversion accuracy high enough, one has to retain as many terms in the expansions as possible. However, this requires that the expansion coefficients be calculated with higher accuracy, which is usually limited by the available computational resources. For nonanalytic and discontinuous functions, the largest errors are expected to accumulate in the vicinity of points of discontinuity. Respectively, the numerical inversion results in a nonphysical peak at small t. On the other hand, asymptotic expansions should be involved at large t.

1.5.1 Shifted Legendre Polynomials

The change of variable $e^{-t} = \zeta$ transforms the interval $(0, \infty)$ of variable t into the interval $(0, 1)$ of variable ζ, and transform (1.27) takes the form

$$F(p) = \int\limits_0^1 f(\zeta)\zeta^{p-1} \mathrm{d}\zeta, \tag{1.29}$$

where $f(\zeta)$ is expressed as a convergent series in terms of polynomials that are orthogonal on the segment $[0, 1]$. The orthogonal polynomials can be chosen in the form of the so-called shifted Legendre polynomials $P_n^*(\zeta)$ [38]:

$$f(\zeta) = \sum_{n=0}^{\infty}(2n+1)a_n P_n^*(\zeta), \quad a_n = \sum_{k=0}^{n}\alpha_k^{(n)}F(k+1), \tag{1.30}$$

where,

$$P_n^*(\zeta) = (-1)^n\sum_{k=0}^{n}\alpha_k^{(n)}\zeta^k, \quad \alpha_k^{(n)} = (-1)^k\binom{n}{k}\frac{(n+k)!}{n!k!}.$$

The coefficients a_n are calculated at a finite number k of points evenly spaced along the real axis of the complex parameter p.

Figures 1.2 and 1.3 demonstrate some results of our calculations performed with 5, 6, and 10 terms taken into account in series (1.30) (the authors are grateful to Dr. V.A. Tkachenko for his assistance in producing these plots). Ten terms proved to be optimal for the approximation with nine significant digits. In all the cases, the accuracy of calculations increases with n.

This algorithm was used to solve a number of problems in the theory of wave propagation and diffraction, including the initial-boundary-value problems for a hydraulic shock [53], pulse impact on an open spherical elastic shell [54], tsunami wave generation [48], etc.

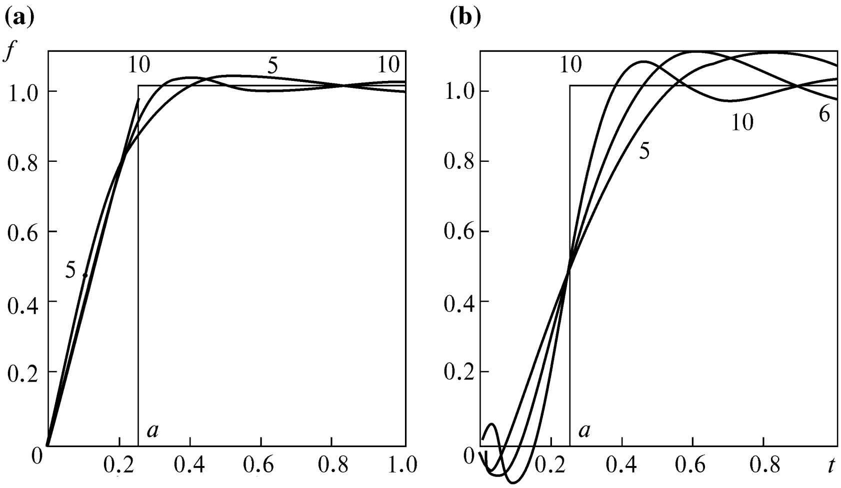

Fig. 1.2 Results of the numerical inversion in terms of shifted Legendre polynomials for **a** linearly increasing function $f(t) = \begin{cases} t/a, & 0 < t \le a, \\ 1, & t > a, \end{cases}$ $F(p) = \frac{1-\exp(-ap)}{ap^2}$ that reduces to a constant, **b** Heaviside step function $f(t) = H(t-a)$, $F(p) = \frac{\exp(-ap)}{p}$

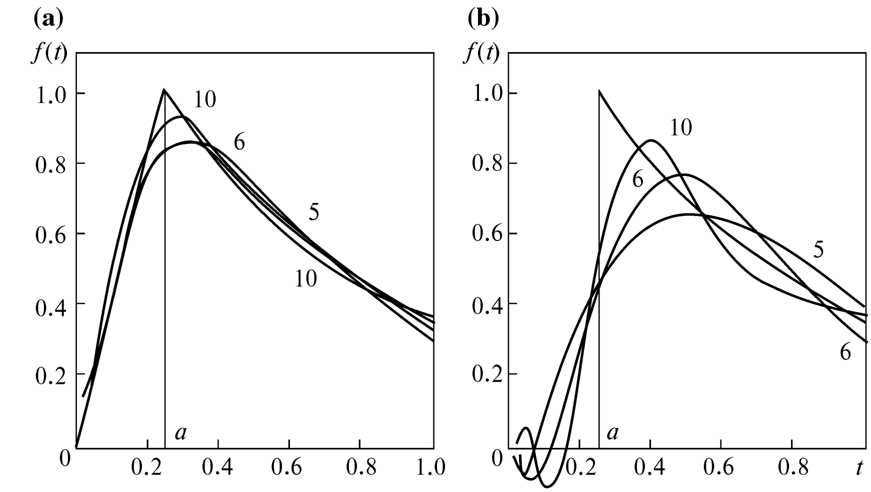

Fig. 1.3 Results of the numerical inversion in terms of shifted Legendre polynomials for **a** linearly increasing and then exponentially decreasing function $f(t) = \begin{cases} t/a, & 0 < t \le a, \\ \exp(-b(t-a)), & t > a, \end{cases}$ $F(p) = \frac{1-\exp(-ap)}{ap^2} + \frac{\exp(-ap)}{p+b} - \frac{\exp(-ap)}{p}$, **b** $f(t) = \exp(-b(t-a))H(t-a)$, $F(p) = \frac{\exp(-ap)}{p-b}$

1.5.2 Fourier Sine Series

We introduce the variable θ instead of time t as

$$\cos\theta = e^{-\sigma t},\ p = (2n+1)\sigma,\ \sigma > 0,\ n = 0, 1, \ldots$$

and write the function $f(t)$ in (1.27) in the form of the sine series [15, 43]:

$$f(t) = f\left(-\frac{1}{\sigma}\ln(\cos\theta)\right) = \Phi(\theta) = \sum_{\nu=0}^{\infty} C_\nu \sin(2\nu+1)\theta. \tag{1.31}$$

Then we obtain a set of linear equations for the coefficients C_ν:

$$C_0 = \frac{4}{\pi}\sigma F(\sigma),\ C_0 + C_1 = \frac{4^2}{\pi}\sigma F(3\sigma),\ 2C_0 + 3C_1 + C_2 = \frac{4^3}{\pi}\sigma F(5\sigma),\ \ldots.$$

Note that the realisation of this algorithm essentially depends on the value of parameter σ and selecting it in the optimal way involves supplementary numerical calculations. Some results of our test calculations are shown in Fig. 1.4.

This algorithm was used to analyse the propagation of acoustic waves in a compressible fluid under the surface wave excitation by bottom movements [2]. The corresponding initial-boundary-value problem is formulated as follows