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Gualtiero Badin
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Variational Formulation of Fluid and Geophysical Fluid Dynamics

Mechanics, Symmetries and Conservation Laws

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In loving memory of Prof. Giuseppe Furlan (1935–2016)

Foreword

In science, as in other walks of life, we are often tempted to do something that will have an immediate impact that seems original and that will garner more funding and get us promoted but that may have little true benefit in the long term. And so we do it, and thereby make a Faustian bargain, not really thinking about the longer term. But that long term might be better served if we could make more of a Proustian bargain in which we remember the accomplishments of the past, search for the meaning in the science, build on secure foundations and so make true advances, even if slowly and intermittently. To proceed this way, we need a proper exposition of those foundations and how they relate to the more applied concerns that we deal with on a daily basis, and it is this noble task that the authors of this book have set themselves. They have returned to the very fundamentals of Geophysical Fluid Dynamics and given us a compelling account of how Hamilton's principle and variational methods provide a secure footing to the subject and give an underlying meaning to its results.

Hamilton's principle provides one of the most fundamental and elegant ways of looking at mechanics. The laws of motion—whether they may be Newton's laws in classical mechanics or the equations of quantum mechanics—emerge naturally by way of a systematic variational treatment from clear axioms. The connection of the conservation properties of the system to the underlying symmetries is made transparent, approximations may be made consistently, and the formulation provides a solid basis for practical applications. In this book, the authors apply this methodology to Geophysical Fluid Dynamics, starting with a derivation of the equations of motion themselves and progressing systematically to approximate equation sets for use with the rapidly rotating and stratified flows that we encounter in meteorology and oceanography. Along the way, we encounter such things as Noether's Theorem, Lagrangian and Eulerian viewpoints, the relabelling symmetry that gives rise to potential vorticity conservation, semi-geostrophic dynamics and the conservation of wave activity. The method also has great practical benefit, for it is only by use of approximate equation sets that we are able to compute the future state of the weather—the lack of the proper use of approximate or filtered equations can be thought of as the cause of the failure of Richardson's heroic effort to numerically predict the weather in 1922, and the proper use of an approximate set was vital for the success of the effort some 30 years later by Charney, Fjortoft and von Neumann, and still today we use approximate equation sets in climate and weather models.

The treatment in the book is unavoidably mathematical, but it is not “advanced”, for it makes use only of fairly standard methods in variational calculus and a little bit of group theory. The book should be accessible to anyone who has such a background although it is not, reusing one of Clifford Truesdell’s many memorable remarks, a mountain railway that will take the reader on a scenic tour of all the famous results with no effort on the reader’s part. But with just a little work, the book will benefit meteorologists and oceanographers who wish to learn about variational methods, and it will benefit physicists and applied mathematicians who wish to learn about Geophysical Fluid Dynamics. And the book reminds us once again that Geophysical Fluid Dynamics is a branch of theoretical physics, as it has always been but as we sometimes forget.

April 2017

Geoffrey K. Vallis
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Preface

The motion of fluids from the smaller to the large scales is described by a complex interplay between the momentum equations and the equations describing the thermodynamics of the system under consideration. The emerging motion comprises several scales, ranging from microscales, to planetary scales, often linked by non-trivial self-similar scalings. At the same time, the motion of classic fluids is described by a specific branch of continuum mechanics. It comes thus natural that one would like to describe the rich phenomenology of the fluid and geophysical fluid motion in a systematic way from first principles, derived by continuum mechanics. One of these first principles is given by Hamilton's principle, which allows to obtain the equations of motion through a variational treatment of the system.

A famous call for the need of a systematic derivation of the equations of Fluid and Geophysical Fluid Dynamics lies in the memorandum sent by the mathematician John von Neumann to Oswald Veblen, written in 1945 and here reported in the Introduction to Chap. 3. The quote reads: "*The great virtue of the variational treatment […] is that it permits efficient use, in the process of calculation, of any experimental or intuitive insight […]. It is important to realize that it is not possible, or possible to a much smaller extent, if one performs the calculation by using the original form of the equations of motion—the partial differential equations. […] Symmetry, stationarity, similitude properties […] applying such methods to hydrodynamics would be of the greatest importance since in many hydrodynamical problems we have very good general evidence of the above-mentioned sort about the approximate aspect of the solution, and the refining of this to a solution of the desired precision is what presents disproportionate computational difficulties […]*" (see reference to von Neumann (1963) of Chap. 3). While sadly von Neumann intended to make practical use of such a treatment to study the aftershocks created by nuclear explosions, the quote still summarizes some of the most important features of the variational method: "*Symmetry, stationarity, similitude properties*". With these properties, von Neumann clearly had in mind the self-similar structure of fluid flows ("*similitude*"), which is indeed the feature that allows us to study different scales of motion through a proper rescaling of the system; he probably had in mind also the study of the stability of the system under consideration ("*stationarity*"); but he mentions also one of the most important results from field-theory that is the study of what he calls with the word "*symmetry*". Continuous symmetries in mechanical systems have in fact the property to be related to conserved quantities, as it is well known by probably the most beautiful theorem in mathematical physics, the celebrated "*Noether's Theorem*". In the specific case of fluid dynamics, the continuum hypothesis is associated to a specific symmetry that is the particle relabelling symmetry. Application of Noether's Theorem results in the fundamental conservation of vorticity in fluids, which is itself linked to the conservation of circulation and of potential vorticity, all quantities that have primary importance in a huge number of applications, ranging from fluids, geophysical fluids, plasmas and astrophysical fluids. It is from the particle relabelling symmetry and Noether's Theorem that one sees that the conservation of vorticity is a fundamental property of the system and does not emerge just from skilful manipulation of the partial differential equations describing the dynamics.

The aim of this book is to go through the development of these concepts.

In Chap. 1, we give a résumé of the aspects of Fluid and Geophysical Fluid Dynamics, starting from the continuum hypothesis and then presenting the governing equations and the conservation of potential vorticity as well as energy and enstrophy, in various approximations.

In Chap. 2, we review the Lagrangian formulation of dynamics starting from Hamilton's Principle of First Action. In the second part of the chapter, Noether's Theorem is presented both for material particles and for continuous systems such as fluids.

In this way, Chap. 1 will serve as an introduction to Fluid and Geophysical Fluid Dynamics to students and researchers of subjects such as physics and mathematics. Chapter 2 will instead serve as an introduction to analytical mechanics to students of applied subjects, such as engineering, climatology, meteorology and oceanography.

In Chap. 3, we first introduce the Lagrangian density for the ideal fluid. The equations of motion are rederived using Hamilton's principle first in the Lagrangian and then in the Eulerian frameworks. The relationship between the two frameworks is thus revealed from the use of canonical transformations. Noether's Theorem is then applied to derive the conservation laws corresponding to the continuous symmetries of the Lagrangian density. Particular attention will be given to the particle relabelling symmetry, and the associated conservation of vorticity.

In Chap. 4, we extend the use of Hamilton's principle to continuously stratified fluids and to uniformly rotating flows. Different sets of approximated equations, which constitute different commonly used approximation in Geophysical Fluid Dynamics, are considered, as well as the form taken by the conservation of potential vorticity in each of them. Finally, the variational methods are applied to study some selected topics of wave dynamics.

Technical derivations of equations that might interrupt the flow of the reading are reported in a number of appendices.

This book should be considered as an elementary introduction. Bibliographical notes at the end of each chapter will guide the reader to more advanced treatments of the subject.

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April 2017

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Gualtiero Badin

Deep gratitude to my wife Isabella who, in the course of our happy marriage, always had to face the unpleasant aspects of daily life alone, so that I was free to devote myself to physics—thank you.

Fulvio Crisciani

Contents

Chapter 1
Fundamental Equations of Fluid and Geophysical Fluid Dynamics

Abstract The motion of fluids from the smaller to the large scales, i.e. until the oceans and atmospheric currents, is described by a complex interplay of the momentum equations and the equations describing the thermodynamics of the specific system. The resulting set of equations constitutes the branch of physics and applied mathematics called Fluid and Geophysical Fluid Dynamics. The continuum hypothesis and the governing equations of Fluid and Geophysical Fluid Dynamics in their inviscid form are here synthetically reviewed. Emphasis is given to the conservation of energy, enstrophy and potential vorticity, which are written in various approximations. The obtained relationships constitute the basis for the development of the following chapters. Chapter 1 aims thus to give only a résumé of the aspects of Fluid and Geophysical Fluid Dynamics which will be considered from the Lagrangian and Hamiltonian point of view in the other chapters. For this reason, several steps in deriving the governing equations are omitted and only the outlines are mostly reported.

Keywords Fluid dynamics · Geophysical fluid dynamics · Ideal fluid · Conservation laws · Rotating flows · Stratified flows · Potential vorticity · Ertel's theorem · Circulation · Shallow water equations · Quasi-geostrophic equations

1.1 Introduction

Fluid dynamics deals with a wide range of scales of motions, ranging from the micro till the planetary scales, and linked by self-similar laws. Once the laws governing the velocity **u**, the pressure p and the density ρ of these fluids are established, and the main goal is to understand, in terms of mathematical models suitably idealized, the rich physical phenomenology exhibited by the fluids. The governing equations are based on the continuous distribution of the fields under consideration. At the larger scales, Geophysical Fluid Dynamics deals with large-scale motions of fluids in the oceans (marine currents) and in the atmosphere (winds), as viewed by a terrestrial observer, i.e. by an observer whose frame of reference is fixed with the Earth. On this subject, Joseph Pedlosky [12] said *"One of the key features of*

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Geophysical Fluid Dynamics is the need to combine approximate forms of the basic fluid-dynamical equations of motion with careful and precise analysis. The approximations are required to make any progress possible, while precision is demanded to make the progress meaningful". In this chapter, this continuum hypothesis and the governing equations are synthetically reviewed in the standard (i.e. nonvariational) approach. While the oceans and the atmosphere are made of viscous, and thus dissipative, fluids, we will here concentrate in the nondissipative equations, in order to allow for a Hamiltonian formulation of the dynamics in the following chapters. Notice that not all the approximations derived in this chapter will be rederived from variational principles in the following chapters and vice versa. Some attention will however be dedicated here to additional approximations, as the nonvariational derivation highlights some physical processes that might be of interest especially for the more applied readers.

1.2 The Continuum Hypothesis

Fluid matter is slippery not only from a practical point of view, but also in the attempt to establish principles and laws of classical physics which govern its evolution. For instance, while the dynamics of a pointlike massive body is, basically, that of Newton's second law, the application of the same equation to a fluid according to the *Lagrangian description* looks problematic as far as an operative definition of a "pointlike fluid body" is not established. In order to attribute the velocity and the acceleration to individual bodies of fluid consistently with Newton's second law, the concept of parcel is introduced and defined as a volume of fluid whose amount is the same at any time. Obviously, a parcel is not pointlike but, rather, it has a finite volume which, in general, changes its shape at each time. Hence, one should pose the question: "How large is a parcel?". The answer goes beyond the simple definition of parcel (in the sense that, a priori, its volume and the related mass could be arbitrary) and relies on the possibility to perform measurements, at the macroscopic scale and in the framework of classical physics, on the fluid, i.e. on its parcels. In a measure process, a probe is put into the fluid and the output of the instrument is a number (usually referred to SI units) which comes from the interaction of the probe with the fluid; this number quantifies a physical property of the fluid. The process can be repeated for each point and time. The volume of fluid to which the instrument responds is much larger than the volume in which variations due to molecular fluctuations take place; in this way, an undesired random variability of the output is avoided. Thus, the volume of a parcel, and of the parcels that interact with the probe, is far larger than the typical distance among the molecules of the fluid. This is a lower bound for the volume of a parcel. On the other hand, the uniqueness of the number provided by the instrument in a single measurement means that, in the interaction with the parcel, the probe feels a uniformly distributed property of the fluid. Thus, *the property of the fluid* of the parcel that interacts with the probe, and hence *of every parcel, is spread uniformly over the volume of the parcel*. This is the *continuum hypothesis*, which ultimately relies on the unicity of the response of a process of measurement within the framework of classical physics. By varying the position of the probe, different outputs are expected and usually obtained. This fact poses an upper bound to the volume of the parcels; in fact, the possibility to detect the variations is associated with spatial distribution of physical quantities whenever the probe interacts with different parcels of the same fluid demands that the volume of the parcels be far smaller of the total volume involved in the measure process.

The same measure process described above can be interpreted from a different point of view, by associating with each position of the probe the related numerical output of the measurement and by assuming the possibility to extend ideally this mapping to the whole volume of fluid under investigation. In this way, named *Eulerian description*, the physical property of the fluid is referred, point by point, to the volume that includes it in terms of a field. Although the Eulerian description is not fit for Newton's second law, it is related to the Lagrangian description by a kinematic constraint according to which the field property at a given location and time must equal the property possessed by the parcel occupying that position on that instant.

In the *Lagrangian description*, instead, the fluid looks like a continuous variety of parcels, so, in principle, each parcel can be identified by a fixed tern of labels and by time. As a parcel does not change the labels in the course of motion, the coordinates of the parcel are a function of the labels and of time; in other words, in the *Lagrangian description*, labels and time are independent variables while the coordinates are variables dependent on the labels and on time. Hence, if a quantity is ascribed to a parcel following the motion, the rate of change of the quantity is simply its differentiation with respect to time. Label coordinates refer to a certain label space, while the space coordinates refer to a certain location space, and the relationship between these two spaces is represented by a nonsingular mapping (in each point of the location space, there is a "labelled" parcel, and in each "labelled" point of the label space, definite space coordinates can be attributed to the parcel occupying that point) whose time evolution describes fluid motion. No criterion to assign the labels is established a priori, provided that each parcel keeps the same labels for all time.

1.3 Derivation of the Equations of Motion

1.3.1 Conservation of Mass

By definition, the mass of any parcel is conserved in time. Thus, if $V(t)$ is the material volume of a certain parcel, then

$$\frac{d}{dt}\int_{V(t)} \rho(\mathbf{r}, t)dV' = 0\,. \tag{1.1}$$

Notice that in the following, the independent variables will sometimes be omitted.

Now, for any scalar θ included in the parcel of volume $V(t)$, the equation

$$\frac{d}{dt}\int_{V(t)} \theta dV' = \int_{V(t)} \left(\frac{D\theta}{Dt} + \theta\ div\ \mathbf{u}\right) dV' \tag{1.2}$$

allows to transfer the time derivative of a space integral inside the space integral itself. A derivation of (1.2) is reported in Appendix A. The velocity $\mathbf{u}$ appearing in the Lagrangian derivative $D/Dt = \partial/\partial t + \mathbf{u} \cdot \nabla$ of (1.2) is the velocity of the parcel. By using (1.2), Eq. (1.1) becomes

$$\int_{V(t)} \left(\frac{D\rho}{Dt} + \rho\ div\ \mathbf{u}\right) dV' = 0\ . \tag{1.3}$$

Because Eq. (1.3) holds for every parcel, the governing equation of the density field

$$\frac{D\rho}{Dt} + \rho\ div\ \mathbf{u} = 0 \tag{1.4}$$

or, equivalently,

$$\frac{\partial \rho}{\partial t} + div(\rho \mathbf{u}) = 0 \tag{1.5}$$

immediately follows.

1.3.2 Incompressibility and Density Conservation

A fluid is said to be incompressible when the density of parcels is not affected by changes in the pressure. Thus, the rate of change of ρ following the motion is zero

$$\frac{D\rho}{Dt} = 0\ . \tag{1.6}$$

In other words, parcels move on trajectories along which the density field takes a constant value. If Eq. (1.6) holds true, then (1.4) implies

$$div\ \mathbf{u} = 0\ , \tag{1.7}$$

which means that the current is solenoidal. In turn, Eq. (1.7) means that every stream tube must be either close, or end on the boundary of the fluid, or extend in a unbounded way in some direction.

1.3.3 Momentum Equation in an Inertial Frame of Reference

Given that the parcel is an individual portion of fluid, in analogy with the Newton's second law for a pointlike massive object, the time derivative of the linear momentum of a parcel is assumed to be equal to the sum of the forces applied to it. The linear momentum of a parcel is defined by $\int_{V(t)} \rho \mathbf{u} dV'$, so its acceleration is given by

$$\frac{d}{dt} \int_{V(t)} \rho \mathbf{u} dV' . \tag{1.8}$$

Unlike a pointlike mass moving without interacting with the surrounding matter, the forces applied to a parcel are not only body forces, such as gravity, but also surface forces due to the interaction of each parcel with those surrounding it. The body forces can be represented by the quantity

$$\int_{V(t)} \rho \mathbf{F} dV' , \tag{1.9}$$

where $\mathbf{F}$ is a force per unit mass that includes gravity acceleration $-g\hat{\mathbf{k}}$, where we have used the standard notation in which $\left(\hat{\mathbf{i}}, \hat{\mathbf{j}}, \hat{\mathbf{k}}\right)$ indicate the orthogonal unit vectors for the (x, y, z) tern. In the following, we will also indicate with $\left(\hat{\mathbf{n}}, \hat{\mathbf{t}}\right)$ the normal and the tangent unit vectors at a certain point of a material surface, respectively. The fundamental surface force mainly comes from the pressure reciprocally exerted at the boundary of the parcels in contact. The parcel included into $V(t)$ experiences the force

$$-\int_{V(t)} \nabla p dV' , \tag{1.10}$$

where $p = p(\mathbf{r}, t)$ is the pressure field. By using (1.8)–(1.10), Newton's second law results in the equation

$$\frac{d}{dt} \int_{V(t)} \rho \mathbf{u} dV' = \int_{V(t)} (\rho \mathbf{F} - \nabla p) \, dV' . \tag{1.11}$$

The l.h.s of (1.11) can be rearranged using (1.2) according to the chain of equalities

$$\begin{aligned} \frac{d}{dt} \int_{V(t)} \rho \mathbf{u} dV' &= \int_{V(t)} \left[\frac{D}{Dt} (\rho \mathbf{u}) + \rho \mathbf{u} \, div \, \mathbf{u} \right] dV' \\ &= \int_{V(t)} \left[\mathbf{u} \frac{D\rho}{Dt} + \rho \frac{D\mathbf{u}}{Dt} + \rho \mathbf{u} \, div \, \mathbf{u} \right] dV' \\ &= \int_{V(t)} \left[\rho \frac{D\mathbf{u}}{Dt} + \mathbf{u} \left(\frac{D\rho}{Dt} + \rho \, div \, \mathbf{u} \right) \right] dV' . \end{aligned} \tag{1.12}$$

Recalling (1.4)–(1.5), Eq. (1.12) simplifies into

$$\frac{d}{dt}\int_{V(t)} \rho \mathbf{u} dV' = \int_{V(t)} \rho \frac{D\mathbf{u}}{Dt} dV' \tag{1.13}$$

so, with the aid of (1.13), Eq. (1.11) becomes

$$\int_{V(t)} \left(\rho \frac{D\mathbf{u}}{Dt} - \rho \mathbf{F} + \nabla p \right) dV' = 0 \,. \tag{1.14}$$

After a trivial rearrangement and the use of position $\mathbf{F} = -g^* \hat{\mathbf{k}}$, Eq. (1.14) yields the so-called *Euler's equation* in the form

$$\frac{D\mathbf{u}}{Dt} = -\frac{\nabla p}{\rho} - g^* \hat{\mathbf{k}} \,. \tag{1.15}$$

Equation (1.15) looks fit for flows of a massive ($-g^* \hat{\mathbf{k}} \neq \mathbf{0}$) but nonrotating Earth or, more realistically, for flows that do not feel Earth's rotation. This point will be clarified at the end of Sect. 1.5.

1.4 Elementary Symmetries of the Euler's Equation

Consider the Euler's equation (1.15) in absence of the body force $\mathbf{F}$,

$$\frac{D\mathbf{u}}{Dt} = -\frac{\nabla p}{\rho} \,. \tag{1.16}$$

Equation (1.16) is invariant under the symmetry transformation

$$\mathbf{r} \to \mathbf{r}' = g_r(\mathbf{r}) \,, \tag{1.17a}$$

$$t \to t' = g_t(t) \,, \tag{1.17b}$$

$$p \to p' = g_p(p) \,, \tag{1.17c}$$

if it satisfies

$$\frac{D\mathbf{u}}{Dt} = -\frac{\nabla p}{\rho} \Rightarrow \frac{D\mathbf{u}'}{Dt'} = -\frac{\nabla' p'}{\rho} \,, \tag{1.18}$$

i.e. if the equations of motion do not change under the transformation (1.17a)–(1.17c). In (1.17a)–(1.17c), $g_r \in G_r$, $g_t \in G_t$, $g_p \in G_p$ are symmetry transformations that belong to the one-parameter groups G_r , G_t, G_p. Notice that the transformation of the pressure field is

$$g_p(p) = p + \bar{p} \quad \text{or} \quad g_p(p) = Cp \,, \tag{1.19}$$

and it is thus determined through the identification of either $\bar{p}$ or the nondimensional constant C so that the transformed pressure field depends on the specific case under consideration and satisfies the invariance of the original equation. In certain cases, g_p will be a function of the space and time coordinates.

For the following symmetries, the theses and proofs will proceed through the statement of the symmetric transformation on the independent variables. The corresponding transformations of the velocity field and on the time and space derivatives are thus determined directly.

1.4.1 Continuous Symmetries

Equation (1.16) satisfies the following continuous symmetries:

1.4.1.1 Gauge Invariance for the Pressure Field

$$g_r(\mathbf{r}) = \mathbf{r} \,, \tag{1.20a}$$

$$g_t(t) = t \,, \tag{1.20b}$$

$$g_p(p) = p + F(t) \,, \tag{1.20c}$$

where $F(t)$ is an arbitrary function of time. Observing that the transformation does not act on the independent variables t, $\mathbf{r}$ and on the dependent variable $\mathbf{u}$, the invariance is trivially proved upon substitution of (1.20a)–(1.20c) in (1.16).

1.4.1.2 Space Translations

$$g_r(\mathbf{r}) = \mathbf{r} + \mathbf{c} \,, \tag{1.21a}$$

$$g_t(t) = t \,, \tag{1.21b}$$

$$g_p(p) = p \,, \tag{1.21c}$$

where $\mathbf{c} \in \mathbb{R}^3$ is a constant vector.

Proof Time differentiation of (1.21a) shows that the transformation does not act on the velocity field, so that $\mathbf{u}' = \mathbf{u}$. Consider, for simplicity and without loss of generality, the space translation in the x direction $x' = x + c$. Because c is a constant,

$$\frac{\partial}{\partial x'} = \frac{\partial}{\partial x} \,. \tag{1.22}$$