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Advances in Variable Structure Systems and Sliding Mode Control—Theory and Applications

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Preface

Variable structure systems (VSS) and its main mode of operation sliding mode control (SMC) are recognized as one of the most efficient tools to deal with uncertain systems due to their robustness and even insensitivity to perturbations [1–3].

The main advantages of VSS/SMC methodology are:

- theoretical insensitivity with respect to the matched perturbations;
- reduced order of sliding mode dynamics;
- finite-time convergence to zero for sliding mode variables.

However, the development of the VSS/SMC theory has shown their main drawbacks: the chattering phenomenon, namely high-frequency oscillations appearing due to the presence of parasitic dynamics of actuators, sensors, and other non-ideality.

During the last decade, one of the main lines in development of the SMC theory was development of the homogeneous higher-order sliding mode controllers (HOSMC) (see [4–6]). At the first stage, the proof of such algorithm was based on the arguments of homogeneity and geometry.

The main driver of development in recent two years is the new Lyapunov-based approaches for HOSMC design and gain selection [7, 8]. Moreover, the development of Lyapunov function approaches allows to design continuous sliding mode algorithms [9–13].

Different properties of SMC algorithms are investigated, like properties of HOSMC for wider classes of homogeneous systems, as well as properties of SMC for stochastic systems [14] and properties of SMC in frequency domain [15, 16]. Different adaptive algorithms were recently developed [17, 18]. These new algorithms were actively used to both ensure the tracking in different control problems and implement it for control in different real-life systems.

This book is an attempt to reflect the recent developments in VSS/SMC theory and reflect the results which are presented. The book consists of three parts: in the first part (i.e., Chaps. 1–7), new VSS/SMC algorithms are proposed and its properties are analyzed; in the second part (i.e., Chaps. 8–13), the usage of VSS/SMC techniques for solutions of different control problems is given; in the third part (i.e., Chaps. 14–16), applications of VSS/SMC to real-time systems are exhibited.

Part I: New VSS/SMC Algorithms and Their Properties (Chaps. 1–7)

In Chap. 1 "Lyapunov-Based Design of Homogeneous High-Order Sliding Modes" by Prof. Jaime A. Moreno, the author provides a Lyapunov-based design of homogeneous high-order sliding mode (HOSM) control and observation (differentiation) algorithms of arbitrary order for a class of single-input-single-output uncertain nonlinear systems. First, the authors recall the standard problem of HOSM control, which corresponds to the design of a state feedback control and an observer for a particular differential inclusion (DI), which represents a family of dynamic systems including bounded matched perturbations/uncertainties. Next, the author provides a large family of zero-degree homogeneous discontinuous controllers solving the state feedback problem based on a family of explicit and smooth homogeneous Lyapunov functions. The author shows the formal relationship between the control laws and the Lyapunov functions. This also gives a method for the calculation of controller gains ensuring the robust and finite-time stability of the sliding set. The required unmeasured states can be estimated robustly and in finite time by means of an observer or differentiator, originally proposed by Prof. A. Levant. The author gives explicit and smooth Lyapunov functions for the design of gains ensuring the convergence of the estimated states to the actual ones in finite time, despite the non-vanishing bounded perturbations or uncertainties acting on the system. Finally, it is shown that a kind of separation principle is valid for the interconnection of the HOSM controller and observer, and the author illustrates the results by means of a simulation on an electromechanical system.

In Chap. 2 "Robustness of Homogeneous and Homogeneizable Differential Inclusions" by Dr. Emmanuel Bernuau, Prof. Denis Efimov, and Prof. Wilfrid Perruquetti, the authors study the problem of robustness of sliding mode control and estimation algorithms with respect to matched and unmatched disturbances. Using the homogeneous theories and locally homogeneous differential inclusions, two sets of conditions are developed to verify the input-to-state stability property of discontinuous systems. The advantage of the proposed conditions is that they are not based on the Lyapunov function method, but more related to algebraic operations over the right-hand side of the system.

In Chap. 3 "Stochastic Sliding Mode Control and State Estimation" by Prof. Alex S. Poznyak, the author deals with the SMC technique applied to stochastic systems affected by additive as well as multiplicative stochastic white noise. The existence of a strong solution to the corresponding stochastic differential inclusion is discussed. It is shown that this approach is workable with the gain control parameter state-dependent on norms of system states. It is demonstrated that under such modification of the conventional SMC, the exponential convergence of the averaged squared norm of the sliding variable to a zone (around the sliding surface) can be guaranteed, of which the bound is proportional to the diffusion parameter in the model description and inversely depending on the gain parameter.

The behavior of a standard super-twist controller under stochastic perturbations is also studied. For system quadratically stable in the mean-squared sense, a sliding mode observer with the gain parameter linearly depending on the norm of the output estimation error is suggested. It has the same structure as deterministic observer based on "the Equivalent Control Method." The workability of the suggested observer is guaranteed for the group of trajectories with the probabilistic measure closed to one. All theoretical results are supported by numerical simulations.

In Chap. 4 "Practical Stability Phase and Gain Margins Concept" by Prof. Yuri Shtessel, Prof. Leonid Fridman, Dr. Antonio Rosales, and Dr. Chandrasekhara Bharath Panathula, the authors present a new concept of chattering characterization for the systems driven by finite-time convergent controllers (FTCC) in terms of practical stability margins. Unmodeled dynamics of order two or more incite chattering in FTCC-driven systems. In order to analyze the FTCC robustness to unmodeled dynamics, the novel paradigm of tolerance limits (TL) is introduced to characterize the acceptable emerging chattering. Following this paradigm, the authors introduce a new notion of Practical Stability Phase Margin (PSPM) and Practical Stability Gain Margin (PSGM) as a measure of robustness to cascade unmodeled dynamics. Specifically, PSPM and PSGM are defined as the values that have to be added to the phase and gain of dynamically perturbed system driven by FTCC so that the characteristics of the emerging chattering reach TL. For practical calculation of PSPM and PSGM, the harmonic balance (HB) method is employed, and a numerical algorithm to compute describing functions (DFs) for families of FTCC (specifically, for nested, and quasi-continuous higher-order sliding mode (HOSM) controllers) was proposed. A database of adequate DFs was developed. A numerical algorithm for solving HB equation using the Newton–Raphson method is suggested to obtain predicted chattering parameters. Finally, computational algorithms to that identify PSPM and PSGM for the systems driven by FTCC were proposed. The algorithm of a cascade linear compensator design that corrected the FTCC, making the values of PSPM and PSGM to fit the prescribed quantities, is suggested. In order to design the flight-certified FTCC for attitude for the F-16 jet fighter, the proposed technique was employed as a case study. The prescribed robustness to cascade unmodeled actuator dynamics was achieved.

In Chap. 5 "On Inherent Gain Margins of Sliding-Mode Control Systems" by Prof. Igor Boiko, the author defines notion of inherent gain margin of sliding mode control systems. It is demonstrated through analysis and examples that an inherent gain margin depends on the sliding mode control algorithm and not on the plant. This property makes the inherent gain margin a characteristic suitable for comparison of different control algorithms. Analysis of the first-order sliding mode, hysteresis relay control, twisting algorithm, and suboptimal algorithm is presented.

In Chap. 6 "Adaptive Sliding Mode Control Based on the Extended Equivalent Control Concept for Disturbances with Unknown Bounds" by Prof. Tiago Roux Oliveira, Prof. José Paulo V.S. Cunha, and Prof. Liu Hsu, the authors propose an adaptive sliding mode framework based on extended equivalent control to deal with disturbances of unknown bounds. Nonlinear plants are considered with a quite general class of (non)smooth disturbances. The proposed adaptation method is able to make the control gain large when the disturbance grows and decrease it if the latter vanishes, allowing for a minimized chattering occurrence. Global stability of the closed-loop system is demonstrated using the proposed adaptive sliding mode control law. Simulations are presented to show the potential of the new adaptation scheme in this adverse scenario of possibly growing or temporarily large disturbances.

In Chap. 7 "Indirect Adaptive Sliding-Mode Control Using the Certainty-Equivalence Principle" by Dr. Alexander Barth, Prof. Markus Reichhartinger, Prof. Kai Wulff, Prof. Johann Reger, Prof. Stefan Koch, and Prof. Martin Horn, the authors address the design of adaptive sliding mode controllers. The presented controllers compensate uncertainties acting on the input channel of the considered system and are characterized by a possible separation into a structured and an unstructured part. The latter class of uncertainty may affect the system in terms of an external disturbance, whereas a structured uncertainty typically occurs in the case of uncertain plant parameters. The presented controller design methodology enhances standard sliding mode controllers by an additional control action generated from an adaptation mechanism. Applying the certainty equivalence principle, it is possible to systematically handle both classes of uncertainties. The controller design is introduced step by step and demonstrated in detail for systems designated to be controlled by the super-twisting algorithm. The deviation of the adaptive part of the controller is thoroughly demonstrated by deriving three different types of adaptation laws. The requirement to enhance sliding mode controllers by the presented adaptive scheme is underpinned by a simulation scenario demonstrating cascaded feedback loops used for speed and current control of a DC motor. Experimental results obtained by a laboratory test-rig consisting of a motor with unbalanced load demonstrate the applicability of the discussed controller design method.

Part II: The Usage of VSS/SMC Techniques for Solutions of Different Control Problems (Chaps. 8–13)

In Chap. 8 "Variable Structure Observers For Nonlinear Interconnected Systems" by Dr. Mokhtar Mohamed, Prof. Xing-Gang Yan, Prof. Sarah K. Spurgeon, and Prof. Zehui Mao, the authors are concentrated on observer design for nonlinear interconnected systems in the presence of nonlinear interconnections and uncertainties. An approach to deal with nonlinear interconnections is proposed by separating the interconnections to linear and nonlinear parts based on an appropriate transformation. Using the structure property of the interconnected systems, novel variable structure dynamics are designed to observe the state variables of the interconnected systems asymptotically with low conservatism. A simulation example and a case study are presented to demonstrate the effectiveness and the feasibility of the developed results.

In Chap. 9 "A Unified Lyapunov Function for Finite Time Stabilization of Continuous and Variable Structure Systems with Resets" by Dr. Harshal B. Oza, Prof. Yury V. Orlov, and Prof. Sarah K. Spurgeon, the authors present a unified

Lyapunov function for finite-time stabilization of continuous and variable structure systems with resets. This chapter aims to uniformly stabilize a perturbed dynamics of the double integrator in the presence of impacts due to the constraints on the position variable. A non-smooth transformation is proposed to first transform the system into a variable structure system that can be studied within the framework of a conventional discontinuous paradigm. Then, a finite-time stable continuous controller is utilized, and stability of the closed-loop dynamics is proven by identifying a new set of Lyapunov functions. The chapter thus contributes to the VSS and SMC theory by the developing mathematical tools for the finite-time stability analysis of such systems in the presence of impacts.

In Chap. 10 "Robustification of Cooperative Consensus Algorithms in Perturbed Multi-Agents Systems" by Prof. Alessandro Pilloni, Prof. Alessandro Pisano, and Prof. Elio Usai, the authors exploit the integral sliding mode design paradigm in the framework of multi-agent systems. Particularly, it is shown how to redesign standard distributed algorithms for estimating the average value and the median value of the agent's initial conditions in spite of perturbations acting on the agent's dynamics. Constructive Lyapunov-based analysis is presented along with simulation results corroborating the developed treatment.

In Chap. 11 "Finite-Time Consensus for Disturbed Multi-Agent Systems with Unmeasured States via Nonsingular Terminal Sliding-Mode Control" by Dr. Xiangyu Wang and Prof. Shihua Li, the authors study the finite-time output consensus problem for leader–follower higher-order multi-agent systems with mismatched disturbances and unmeasured states. This problem is solved by using a feedforward–feedback composite control method which combines the integral-type non-singular terminal sliding mode control approach and a finite-time observer technique together. The main contributions include three aspects: Firstly, in the presence of mismatched disturbances and unmeasured states, the finite-time output consensus is realized by utilizing the distributed active anti-disturbance control for the first time. Secondly, the results extend the applicable scope of the distributed active anti-disturbance control from state feedback to output feedback. Thirdly, the disturbances considered in this chapter are allowed to be faster time-varying or have higher-order forms, which are not limited to slow time-varying types any more.

In Chap. 12 "Discrete Event-Triggered Sliding Mode Control" by Prof. Abhisek K. Behera and Prof. Bijnan Bandyopadhyay, the authors present a discrete event-triggered SMC strategy for linear systems. Generally, in the event-triggered control, the state is continuously monitored to generate the possible triggering instant, which may incur additional cost and complexity. To overcome this, a discrete event-triggered SMC is proposed which evaluates event periodically and also guarantees the robust performance of the system. The discrete-time SMC is designed considering the triggering rule that ensures the stability with the discrete event-triggering strategy.

In Chap. 13 "Fault Tolerant Control Using Integral Sliding Modes" by Prof. Christopher Edwards, Dr. Halim Alwi, and Dr. Mirza Tariq Hamayun, the authors consider so-called integral sliding modes (ISM) and demonstrate how they can be employed in the context of fault-tolerant control. Two distinct classes of problems are considered: Firstly, a fault-tolerant ISM controller is designed for an over-actuated linear system; secondly, an ISM scheme is retrofitted to an existing feedback control scheme for an over-actuated uncertain linear system with the objective of retaining the preexisting nominal performance in the face of faults and failures. The chapter includes with a case study describing the implementation of an LPV extension of one of the ISM schemes on a motion simulator configured to represent a Boeing 747 aircraft subject to realistic fault scenarios.

Part III: Applications of VSS/SMC to Real-Time Systems (Chaps. 14–16)

In Chap. 14 "Speed Control of Induction Motor Servo Drives Using Terminal Sliding-Mode Controller" by Prof. Yong Feng, Dr. Minghao Zhou, Prof. Fengling Han, and Prof. Xinghuo Yu, the authors apply a non-singular terminal sliding mode control method for the servo system of induction motors. The non-singular terminal sliding mode controllers for speed, flux, and currents are presented, respectively. The switching signals in the controller are softened to generate the continuous output signals of the controllers using the equivalent low-pass filters. Therefore, both the chattering is attenuated and the singularity is eliminated, which means that the controllers can be used in the practical applications.

In Chap. 15 "Sliding Modes Control in Vehicle Longitudinal Dynamics Control" by Prof. Antonella Ferrara and Dr. Gian Paolo Incremona, the authors present recent developments produced at the University of Pavia on application of sliding mode control to the automotive field. Specifically, the chapter focuses on the use of advanced SMC schemes to efficiently solve traction control and vehicle platooning control problems. A slip ratio SMC scheme is described, analyzed, and assessed in simulation. Then, the vehicle platooning control problem is introduced as an extended case of the previously described problem. A vehicle longitudinal dynamics control scheme, based on a suboptimal second-order SMC, is presented and coupled with the slip rate control scheme which allows to generate the correct traction control. The validation in simulation on a realistic scenario of the overall scheme is also discussed.

In Chap. 16 "Sliding Mode Control of Power Converters with Switching Frequency Regulation" by Dr. Víctor Repecho, Dr. Domingo Biel, Dr. Josep M. Olm, and Prof. Enric Fossas, the authors introduce a hysteresis band control loop that provides fixed switched frequency in sliding mode controlled systems while keeping the beneficial properties of sliding motion. The proposal is exemplified in DC-to-DC and DC-to-AC power converters carrying out regulation and tracking tasks, respectively, in the face of load disturbances and input voltage variations.

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Part I
New VSS/SMC Algorithms and Their Properties

Chapter 1
Lyapunov-Based Design of Homogeneous High-Order Sliding Modes

Jaime A. Moreno

1.1 Introduction

Sliding Mode (SM) Control (SMC) [65, 66] aims at designing a sliding variable σ and to force it to $\sigma \equiv 0$ in finite time and to keep it in zero for all future times despite uncertainties and perturbations. For this it is required a discontinuous control action. Classical (or First Order (FO)) SMC achieves this objective when the sliding variable has relative degree $\rho = 1$ with respect to the control variable. Higher Order Sliding Mode (HOSM) Control [24, 38, 40, 41, 43, 65] extends these results to sliding variables σ with arbitrary relative degree $\rho > 1$. Since the implementation of a SMC requires the values of the sliding variable and all its derivatives up to order $\rho - 1$, i.e. $\sigma(t), \dot{\sigma}(t), \ldots, \sigma^{(\rho-1)}(t)$, in HOSMC it has been necessary to develop HOSM Differentiators [4, 5, 16, 20, 22, 35, 37, 39, 64] capable of estimating these derivatives of the sliding variable also in finite time and despite of the uncertainties and perturbations present in the system. These Exact Differentiators make also use of discontinuous output injection to achieve this goal, since smooth observers or differentiators are not able to achieve the objective in the presence of non vanishing uncertainties/perturbations. Since the classical FOSMC does not require any derivative of the sliding variable to be implemented, the Exact Differentiators are a particular development of HOSM's.

Due to the uncertainties and perturbations present in the system, the description of the dynamics of the ρ sliding variables $\sigma, \dot{\sigma}, \ldots, \sigma^{(\rho-1)}$ is naturally described not by a differential equation (DE) but by a Differential Inclusion (DI). One of the main tasks of SMC consists in designing an appropriate sliding variable σ. The sliding variable σ is selected in such a way that the reduced dynamics living on the sliding

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set $\sigma(t) \equiv 0,\ \dot{\sigma}(t) \equiv 0, \ldots, \sigma^{(\rho-1)}(t) \equiv 0$ has the desired behavior, as e.g. it has a robust and asymptotically stable equilibrium point. And thus the main problem of HOSMC reduces to the Finite Time (FT) stabilization of the sliding set for the DI describing its behavior, and this task includes also the FT estimation of the sliding variables.

One of the main ingredients of SMC, the discontinuous control action, becomes also its main disadvantage for the applications: forcing a sliding mode induces a high frequency switching of the control variable and this produces the infamous "chattering" effect, which has undesirable effects as reducing the life of the actuators and exciting high frequency dynamics of the system. HOSM Control helps in mitigating the chattering effect, because introducing extra integrators in the control, and therefore increasing artificially the relative degree of the plant, a continuous (or even smooth) control action can be achieved, at the cost of a higher order sliding set. A further benefit of SMC is the order reduction of the plant's dynamics, since the main design work has to be done on the (reduced) dynamics living on the sliding set. Classical FOSMC allows a reduction of only one dimension (in the Single Input case) while HOSMC permits a reduction of an arbitrary number of degrees up to the order of the system.

(Weighted) Homogenous Differential Equations (HDE) are a very special class of nonlinear systems having very nice and simple properties [2, 3, 10, 28]. For example, for homogeneous systems: (i) local attractivity is equivalent to global asymptotic stability, (ii) internal stability of a system with inputs is equivalent to external stability, (iii) asymptotic stability with negative degree of homogeneity is equivalent to FT stability, etc. These nice properties are also valid for DI's [6, 8, 9, 40, 44]. The FT stabilization and the FT and exact estimation of the sliding variables of the DI describing them requires discontinuous control actions in the controller and discontinuous injection terms in the observer (differentiator). The design and analysis of the robustness, accuracy and convergence properties of the discontinuous controller and observer becomes much simpler if the homogeneity property is imposed on the controlled system and on the estimation error of the observer. This explains that homogeneity has become the main ingredient of HOSMs: essentially all HOSM controllers and observers designed up to now are homogenous.[1]

In fact, the design of discontinuous (and so called quasi-continuous) HOSM controllers and differentiators has been based on geometric methods (which are usually effective for low order or low relative degree systems) [36–39] or, more recently, on the use of Homogeneity and contraction properties of Differential Inclusions [40–43]. It is precisely the homogeneity [3, 8, 40] the property allowing to establishing basic *qualitative* properties of homogeneous HOSM algorithms, as e.g. globality, finite-time convergence, robustness and the type of accuracy.

In contrast, Modern Control Theory is based on the use of Lyapunov or Lyapunov-like Functions (LF) for analysis and design [23, 50]. This is due to the tight connections of this formalism with optimal control, robustness and the diverse internal

[1]For FOSM homogeneity does not play an important role.

and external stability concepts [23, 50]. In particular, for the design of feedback controllers the concept of (Robust) Control Lyapunov Functions (CLFs) has played a major role in the development of control design methods in the last twenty five years [23]. In particular, Classical SMC design is based on the use of Lyapunov functions. One advantage of LF's is that they provide *quantitative* measures which are helpful in the design of controllers and observers. Many of the modern and numerically effective design methods have at their core LF's, as e.g. LMI's for linear and nonlinear systems. It is therefore a natural idea to try to combine homogeneity with a Lyapunov-based design to enhance the modern HOSM control theory, rendering it more quantitative.

In recent years some efforts have been devoted to build explicit smooth and non-smooth, weak and strong Lyapunov functions for some Second Order Sliding Mode (SOSM) Controllers and observers, such as Twisting and Super-Twisting Algorithms [45, 47–49, 54–57]. In [46] a Lyapunov-based design of an output feedback controller, comprising homogeneous SOSM controller and observer, has been obtained. For HOSMC in [27] the authors use the basic idea of the Lyapunov redesign [34] to render a nominal finite-time convergent controller, as e.g. those proposed in [2, 30, 31, 61], and for which a Lyapunov function is already known, robust against matched and bounded perturbations by means of an extra discontinuous control. Unfortunately, the resulting closed loop system is not homogeneous, so that it does not have the nice properties of classical HOSMC [40].

Homogeneous HOSM controllers of arbitrary order based on (explicit) Lyapunov functions were obtained for the first time in [12] (see also [11, 13, 15]) while for arbitrary order HOSM differentiators explicit and smooth Lyapunov Functions were obtained in [14]. Very recently, a new family of so called relay or quasi-continuous polynomial HOSM controllers has been introduced in [18, 19], and a Lyapunov function is obtained for some relay polynomial cases, but no Lyapunov approach is developed for the HOSM differentiator. In [58, 59] Finite-Time convergent controllers have been designed by means of *implicit* Lyapunov functions (ILF), and (quasi-continuous) HOSM controllers can be obtained if some Matrix Inequalities are fulfilled. However, the quasi-continuous controller is also implicitly defined, so that for its implementation the Lyapunov function has to be calculated on-line. The ILF method provides only quasi-continuous controllers, and it has not been yet possible to design exact HOSM differentiators using ILFs.

The main purpose of this chapter is to present some recent advances towards developing a Lyapunov-based approach to the design of homogeneous HOSM control and observation. We develop *explicit* LFs for HOSM controllers and Observers (Differentiators) for the DI describing the dynamics of the sliding variables in HOSM. The use of Lyapunov functions provides a procedure for the gain tuning of the HOSM controllers and observers, it allows the estimation of the convergence time; and it permits the extension to variable-gain discontinuous and quasi-continuous HOSM controllers. Our results are inspired by and constitute a generalization to the discontinuous case of the results for continuous systems [2, 30, 31, 61, 62, 68–70].

The rest of the chapter is organized as follows. In the Sect. 1.2, we give some preliminaries on homogeneous functions and systems. In Sect. 1.3 we formulate the (standard HOSM) problem to be solved. Section 1.4 presents the Lyapunov-based HOSM controllers along with the explicit Lyapunov Functions associated to them. Section 1.5 presents the proofs of the main results of the previous section, and it can be skipped from the first reading without loosing the main track of the ideas. In Sect. 1.6 we show that for discontinuous, homogeneous differentiators there exist smooth LFs for the differentiators for appropriate values of the gains. The proof of this important fact is given in Sect. 1.7. Although this has been shown for the first time in the discontinuous case in [14] we provide here a different Lyapunov function that allows to design the gains of the differentiator independently of the order. In Sect. 1.8 it is shown that the combination of a homogeneous HOSM controller with a homogeneous HOSM observer leads to a globally FT stable output feedback controller, and that a kind of separation principle is available in the global case. Section 1.9 presents some numerical results and in Sect. 1.10 we draw some final comments and conclusions. The Appendix "Some Technical Lemmas on Homogeneous Functions" contains some technical results.

1.2 Preliminaries: Differential Inclusions and Homogeneity

We recall some important concepts about DI's, homogeneity and homogeneous DI's [2, 3, 6–10, 17, 21, 29, 40, 44], which are used in the chapter.

Uncertain or discontinuous systems are more appropriately described by Differential Inclusions (DI) $\dot{x} \in F(t, x)$ than by Differential Equations (DE). A solution of this DI is any function $x(t)$, defined in some interval $I \subseteq [0, \infty)$, which is absolutely continuous on each compact subinterval of I and such that $\dot{x}(t) \in F(t, x(t))$ almost everywhere on I. Thus, for a discontinuous DE $\dot{x} = f(t, x)$ the function $x(t)$ is said to be a generalized solution of the DE if and only if it is a solution of the associated DI $\dot{x} \in F(t, x)$. We will consider the DI $\dot{x} \in F(t, x)$ associated to $\dot{x} = f(t, x)$, as the one given by the approach of A.F. Filippov [3, 21, Sect. 1.2]. So, we refer to such DI as Filippov DI and to its solutions as Filippov solutions.

The multivalued map $F(t, x)$ satisfies the *standard assumptions* if: (H1) $F(t, x)$ is a nonempty, compact, convex subset of $\mathbb{R}^n$, for each $t \geq 0$ and each $x \in \mathbb{R}^n$; (H2) $F(t, x)$ as a set valued map of x, is upper semi-continuous for each $t \geq 0$; (H3) $F(t, x)$ as a set valued map of t, is Lebesgue measurable for each $x \in \mathbb{R}^n$. (H4) $F(t, x)$ is locally bounded. Recall that a set valued map $G : \mathbb{R}^{n_1} \rightrightarrows \mathbb{R}^{n_2}$ with compact values is *upper-semicontinuous* if for each x_0 and for each $\varepsilon > 0$ there exists $\delta > 0$ such that $G(x) \subseteq G(x_0) + B_\varepsilon$, provided that $x \in B_\delta(x_0)$. It is well-known that, see [21] or [3, Theorem 1.4], if the multivalued map $F(t, x)$ satisfies the standard assumptions then for each pair $(t_0, x_0) \in [0, \infty) \times \mathbb{R}^n$ there is an interval I and at least a solution $x(t) : I \to \mathbb{R}^n$ such that $t_0 \in I$ and $x(t_0) = x_0$. A DI $\dot{x} \in F(x)$ (a DE $\dot{x} = f(x)$) is called *globally uniformly finite-time stable* (GUFTS) at 0, if $x(t) = 0$ is a Lyapunov-stable solution and for any $R > 0$ there exists $T > 0$ such that the trajectory starting within the ball $\|x\| < R$ reaches zero in the time T.

Continuous and discontinuous homogeneous functions and systems have a long history [2, 3, 6, 8–10, 25, 28, 40, 44, 51–54, 71]. We recall this important property. For a given vector $x = [x_1, \ldots, x_n]^\top \in \mathbb{R}^n$ and for every $\varepsilon > 0$, the dilation operator is defined as $\Delta_\varepsilon^r x := [\varepsilon^{r_1} x_1, \ldots, \varepsilon^{r_n} x_n]^\top$, where $r_i > 0$ are the weights of the coordinates, and let $\mathbf{r} = [r_1, \ldots, r_n]^\top$ be the vector of weights. A function $V : \mathbb{R}^n \to \mathbb{R}$ (respectively, a vector field $f : \mathbb{R}^n \to \mathbb{R}^n$, or a vector-set field $F(x) \subset \mathbb{R}^n$) is called $\mathbf{r}$-homogeneous of degree $l \in \mathbb{R}$ if the identity $V(\Delta_\varepsilon^r x) = \varepsilon^l V(x)$ holds for every $\varepsilon > 0$ (resp., $f(\Delta_\varepsilon^r x) = \varepsilon^l \Delta_\varepsilon^r f(x)$, or $F(\Delta_\varepsilon^r x) = \varepsilon^l \Delta_\varepsilon^r F(x)$). Along this paper we refer to this property as $\mathbf{r}$-*homogeneity* or simply *homogeneity*. A system is called homogeneous if its vector field (or vector-set field) is $\mathbf{r}$-homogeneous of some degree.

Given a vector $\mathbf{r}$ and a dilation $\Delta_\varepsilon^r x$, the homogeneous norm is defined by $\|x\|_{\mathbf{r},p} := \left(\sum_{i=1}^n |x_i|^{\frac{p}{r_i}}\right)^{\frac{1}{p}}$, $\forall x \in \mathbb{R}^n$, for any $p \geq 1$, and it is an $\mathbf{r}$-homogeneous function of degree 1. The set $S = \{x \in \mathbb{R}^n : \|x\|_{\mathbf{r},p} = 1\}$ is the corresponding homogeneous unit sphere. The following Lemma provides some important properties of homogeneous functions and vector fields (some others are recalled in the Appendix).

Lemma 1.1 ([3, 10, 29]) *For a given family of dilations $\Delta_\varepsilon^r x$, and continuous real-valued functions V_1, V_2 on $\mathbb{R}^n$ (resp., a vector field f) which are* $\mathbf{r}$*-homogeneous of degrees $m_1 > 0$ and $m_2 > 0$ (resp., $l \in \mathbb{R}$), we have:*

(i) $V_1 V_2$ is homogeneous of degree $m_1 + m_2$.

(ii) For every $x \in \mathbb{R}^n$ and each positive-definite function V_1, we have $c_1 V_1^{\frac{m_2}{m_1}}(x) \leq V_2(x) \leq c_2 V_1^{\frac{m_2}{m_1}}(x)$, where $c_1 \triangleq \min_{\{z:V_1(z)=1\}} V_2(z)$ and $c_2 \triangleq \max_{\{z:V_1(z)=1\}} V_2(z)$. Moreover, if V_2 is positive definite, there exists $c_1 > 0$.

(iii) $\partial V_1(x)/\partial x_i$ is homogeneous of degree $m_1 - r_i$, with r_i being the weight of x_i.

(iv) The Lie's derivative of $V_1(x)$ along $f(x)$, $L_f V_1(x) := \frac{\partial V_1(x)}{\partial x} \cdot f(x)$, is homogeneous of degree $m_1 + l$.

It is worth to recall that for homogeneous systems the local stability implies global stability and if the homogeneous degree is negative asymptotic stability implies finite-time stability [3, 8, 40, 44]. (Asymptotic) stability of homogeneous systems and homogeneous DI's can be studied by means of homogeneous LFs (HLFs), see for example [3, 6, 8–10, 25, 28, 40, 44, 51, 63, 71]: Assume that the origin of a homogeneous Filippov DI $\dot{x} \in F(x)$ is strongly globally AS. Then, there exists a $\mathscr{C}^\infty$ homogeneous strong LF.

The following robustness result of asymptotically stable homogeneous Filippov Differential Inclusions is of paramount importance for the assertion of the accuracy properties of HOSM algorithms in presence of measurement or discretization noise or also delay and external perturbations. They have been established by Levant [8, 39, 40, 44].

Theorem 1.1 *Let $\dot{x} \in F(x)$ be a globally uniformly finite-time stable homogeneous Filippov inclusion with homogeneity weights $\mathbf{r} = (r_1, \ldots, r_n)$ and degree $l < 0$, and let $\tau > 0$. Suppose that a continuous function $x(t)$ is defined for any $t \geq -\tau^l$ and* *satisfy some initial conditions* $x(t) = \xi(t)$ *for* $t \in [-\tau^l, 0]$. *Then if* $x(t)$ *is a solution of the perturbed differential inclusion*

$$\dot{x}(t) \in F_\tau\left(x\left(t + \left[-\tau^l, 0\right]\right)\right), \; 0 < t < \infty,$$

then the inequalities $|x_i| < \gamma_i \tau^{r_i}$ *are established in finite time with some positive constants* γ_i *independent of* τ *and* ξ.

Along this paper we use the following notation. For a real variable $z \in \mathbb{R}$ and a real number $p \in \mathbb{R}$ the symbol $\lceil z \rfloor^p = |z|^p sign[z]$ is the sign preserving power p of z. According to this $\lceil z \rfloor^0 = \mathrm{sign}[z]$, $\frac{\mathrm{d}}{\mathrm{d}z} \lceil z \rfloor^p = p\,|z|^{p-1}$ and $\frac{\mathrm{d}}{\mathrm{d}z} |z|^p = p \lceil z \rfloor^{p-1}$ almost everywhere for z. Note that $\lceil z \rfloor^2 = |z|^2 \mathrm{sign}[z] \neq z^2$, and if p is an odd number then $\lceil z \rfloor^p = z^p$ and $|z|^p = z^p$ for any even integer p. Moreover, $\lceil z \rfloor^p \lceil z \rfloor^q = |z|^{p+q}$, $\lceil z \rfloor^p \lceil z \rfloor^0 = |z|^p$, and $\lceil z \rfloor^0 |z|^p = \lceil z \rfloor^p$. We also use the following notation: For a vector $x \in \mathbb{R}^n$ we denote by $\bar{x}_i \in \mathbb{R}^i$ the vector of the first i components, i.e. $\bar{x}_i = [x_1, \ldots, x_i]^T$. Similarly, we denote by $\underline{x}_i \in \mathbb{R}^{n-(i-1)}$ the vector of the last components, i.e. $\underline{x}_i = [x_i, \ldots, x_n]^T$. Note that $x = \bar{x}_n = \underline{x}_1$ are equivalent.

1.3 SISO Regulation and Tracking Problem

Consider a SISO dynamical system affine in the control

$$\dot{z} = f(t, z) + g(t, z)\,u\,, \; y = h(t, z)\,, \tag{1.1}$$

where $z \in \mathbb{R}^n$ defines the state vector, $u \in \mathbb{R}$ is the control input, $y \in \mathbb{R}$ is the output and $h(t, z) : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ is a smooth output function. A standard problem of control is the output tracking problem [32], consisting in forcing the output y to track a (time-varying) signal $y_R(t)$. Usually this problem has associated a (robust) disturbance decoupling or attenuation property [32, 33]. For our purposes we assume that the functions $f(t, z)$ and $g(t, z)$ are uncertain smooth vector fields on $\mathbb{R}^n$ and the dimension n can also be unknown. The control objective, i.e. the standard HOSM control problem [38, 40, 65], consists in making the output $\sigma = y - y_R$ vanishes in finite time and to keep $\sigma \equiv 0$ exactly by a bounded (discontinuous) feedback control. All differential equations are understood in the Filippov's sense [21].

When the relative degree ρ with respect to σ is known, well defined and constant this is equivalent to designing a controller for the DI

$$\sum\nolimits_{DI} : \begin{cases} \dot{x}_i = x_{i+1}, \; i = 1, \ldots, \rho - 1, \\ \dot{x}_\rho \in [-C, C] + [K_m, K_M]u\,, \end{cases} \tag{1.2}$$

where $x = (x_1, \ldots, x_\rho)^T = (\sigma, \dot{\sigma}, \ldots, \sigma^{(\rho-1)})^T$ and $\sigma^{(i)} = \frac{d^i}{dt^i} h(z, t)$. Note that Σ_{DI} does not depend on the particular properties of the original systems' dynamics and the DI only retains the constants ρ, C, K_m and K_M. Due to the persisting uncertainty/perturbation causing the constant $C > 0$ the stabilization of $x = 0$ requires a control discontinuous at $x = 0$, and therefore the classical nonlinear control techniques, that aim at designing a continuous controller as e.g. [32–34], cannot be applied.

For homogenous HOSM [40] the problem is solved by designing a bounded memoryless feedback $\mathbf{r}$-homogeneous control law of degree 0 (called also ρ-sliding homogeneous)

$$u = \varphi\left(x_1,\, x_2, \ldots, x_\rho\right) = \varphi\left(\varepsilon^{r_1} x_1,\, \varepsilon^{r_2} x_2, \ldots, \varepsilon^{r_\rho} x_\rho\right)\,,\ \forall \varepsilon > 0\,, \tag{1.3}$$

with $\mathbf{r} = (\rho, \rho - 1, \ldots, 1)$, that renders the origin $x = 0$ finite-time stable for Σ_{DI}. The motion on the set $x = 0$, which consists of Filippov trajectories [21], is called an *ρth-order sliding mode*. The function φ is discontinuos at the ρ-sliding set ($x = 0$). The closed-loop inclusion (1.2)–(1.3) is an $\mathbf{r}$-homogeneous Filippov DI of degree -1 satisfying *standard assumptions*. In the next Sect. 1.4 we provide some families of homogeneous HOSM controllers that solve the problem for any set of parameters $(\rho,\ C,\ K_m,\ K_M)$. They are similar to the ones proposed by A. Levant but are characterized by the fact that they are obtained by means of explicit smooth (Control) Lyapunov Functions.

Since the implementation of the controller (1.3) requires the values of σ and its derivatives up to $\sigma^{(\rho-1)}$, i.e. the state x of Σ_{DI}, we provide in Sect. 1.6 a homogeneous HOSM observer, able to estimate in finite time and robustly the states of Σ_{DI} for any set of parameters $(\rho,\ C,\ K_m,\ K_M)$. Again this observer corresponds to Levant's robust and exact differentiator [39, 40], but our results are distinguished by the fact that the design is based on explicit homogeneous smooth Lyapunov functions.

Finally we note that if the control enters the system (1.1) non affinely the problem can be reduced to the affine form by introducing an integrator and extending the relative degree to $\rho + 1$.

1.4 Lyapunov Based HOSM Controllers

In a series of (by now) classical works, and using basically geometric tools and homogeneous differential equations, A. Levant has derived some families of homogeneous Second and Higher Order Sliding Mode Controllers [36, 38–43]. Recently, in [18, 19] (see also [11–13, 18, 19]), he has obtained also Lyapunov functions for some "relay polynomial" controllers.

Based on smooth (Control) Lyapunov functions we derive a full family of homogeneous HOSM Controllers, which are different from Levant's families (see [11–13]). Given the relative degree $\rho \geq 2$ we assign the homogeneity weights $r_i = \rho - i + 1$ to the variables x_i, obtaining the vector $\mathbf{r} = (\rho,\ \rho - 1, \ldots,\ 1)$, and we define an arbitrary non-decreasing sequence of positive real numbers α_i, so that $0 \leq \alpha_1 \leq \cdots \leq \alpha_{\rho-1} \leq \alpha_\rho$. Furthermore, we define recursively, for $i = 2, \ldots, \rho$, the $\mathscr{C}^1$ **r**-homogeneous functions

$$\sigma_1(\bar{x}_1) = \lceil x_1 \rfloor^{\frac{\rho+\alpha_1}{\rho}}, \ldots, \sigma_i(\bar{x}_i) = \lceil x_i \rfloor^{\frac{\rho+\alpha_i}{\rho-i+1}} + k_{i-1}^{\frac{\rho+\alpha_i}{\rho-i+1}} \lceil \sigma_{i-1}(\bar{x}_{i-1}) \rfloor^{\frac{\rho+\alpha_i}{\rho+\alpha_{i-1}}}, \tag{1.4}$$

with constants $k_i > 0$. Recall that $\bar{x}_i = [x_1, \ldots, x_i]^T$.

For any constant $m \geq \max_{1 \leq i \leq \rho} \{2\rho + 1 + \alpha_{i-1} - i\}$ we also define recursively, for $i = 2, \ldots, \rho$, the $\mathscr{C}^1$ **r**-homogeneous functions

$$V_1(x_1) = \frac{\rho}{m} |x_1|^{\frac{m}{\rho}}, \ldots, V_i(\bar{x}_i) = \gamma_{i-1} V_{i-1}(\bar{x}_{i-1}) + W_i(\bar{x}_i) \tag{1.5}$$

$$W_i(\bar{x}_i) = \frac{r_i}{m} |x_i|^{\frac{m}{r_i}} - \lceil \nu_{i-1}(\bar{x}_{i-1}) \rfloor^{\frac{m-r_i}{r_i}} x_i + \left(1 - \frac{r_i}{m}\right) |\nu_{i-1}(\bar{x}_{i-1})|^{\frac{m}{r_i}}, \tag{1.6}$$

$$\nu_1(x_1) = -k_1 \lceil \sigma_1 \rfloor^{\frac{r_2}{\rho+\alpha_1}} = -k_1 \lceil x_1 \rfloor^{\frac{\rho-1}{\rho}}, \ldots, \nu_i(\bar{x}_i) = -k_i \lceil \sigma_i(\bar{x}_i) \rfloor^{\frac{r_{i+1}}{\rho+\alpha_i}}, \tag{1.7}$$

with (arbitrary) constants $\gamma_i > 0$. $\sigma_i(\bar{x}_i)$, $\nu_i(\bar{x}_i)$ and $V_i(\bar{x}_i)$ are **r**-homogeneous of degrees $\rho + \alpha_i$, r_{i+1} and m, respectively. As it will be shown in Sect. 1.5 $V_c(x) = V_\rho(\bar{x}_\rho)$ is a smooth (Control) Lyapunov Function for the uncertain plant (1.2).

From $V_c(x)$ we can derive different controllers for (1.2). In particular, we obtain the following family of Discontinuous and Quasi-continuous controllers

$$u_D = -k_\rho \varphi_D(x) = -k_\rho \lceil \sigma_\rho(x) \rfloor^0, \tag{1.8}$$

$$u_Q = -k_\rho \varphi_Q(x) = -k_\rho \frac{\sigma_\rho(x)}{M(x)}, \tag{1.9}$$

where $M(x)$ is any continuous **r**-homogeneous positive definite function of degree $\rho + \alpha_\rho$, and (for simplicity) we assume that it is scaled so that $\left|\frac{\sigma_\rho(x)}{M(x)}\right| \leq 1$. The homogeneous controllers (1.8)–(1.9) are derived from $V_c(x) = V_\rho(\bar{x}_\rho)$ by imposing the condition $\frac{\partial V_c(x)}{\partial x_\rho} \varphi(x) > 0$ at all points where $\frac{\partial V_c(x)}{\partial x_\rho} \neq 0$.

The values of k_i, for $i = 1, \ldots, \rho - 1$ can be fixed depending only on ρ and α_i, they are the same for the Discontinuous and the Quasi-continuous controllers, and they are independent of K_m, K_M and C. k_ρ in contrast is selected depending on the values K_m, K_M and C to induce the **r**th order sliding mode, and they are different for the Discontinuous and the Quasi-continuous controllers. Discontinuous controllers are discontinuous not only on the sliding set $\{x = 0\}$ but also when $\sigma_\rho(x) = 0$, while Quasi-Continuous controllers are discontinuous only on the sliding set. Due to this fact they produce less chattering.

Depending on the selection of the free parameters $0 \leq \alpha_1 \leq \cdots \leq \alpha_\rho$ we obtain different families of controllers. We illustrate them presenting the controllers of orders $\rho = 2, 3, 4$:

- Discontinuous Controllers
  - Nested Sliding Controllers: when some of the α_i are different

$$u_{2D} = -k_2 \left[\lceil x_2 \rfloor^{2+\alpha_2} + k_1^{2+\alpha_2} \lceil x_1 \rfloor^{\frac{2+\alpha_2}{2}} \right]^0$$

$$u_{3D} = -k_3 \left[\lceil x_3 \rfloor^{3+\alpha_3} + k_2^{3+\alpha_3} \left[\lceil x_2 \rfloor^{\frac{3+\alpha_2}{2}} + k_1^{\frac{3+\alpha_2}{2}} \lceil x_1 \rfloor^{\frac{3+\alpha_2}{3}} \right]^{\frac{3+\alpha_3}{3+\alpha_2}} \right]^0 \quad (1.10)$$

$$u_{4D} = -k_4 \left[\lceil x_4 \rfloor^{4+\alpha_4} + k_3^{4+\alpha_4} \left[\lceil x_3 \rfloor^{\frac{4+\alpha_3}{2}} + k_2^{\frac{4+\alpha_3}{2}} \left[\lceil x_2 \rfloor^{\frac{4+\alpha_2}{3}} + k_1^{\frac{4+\alpha_2}{3}} \lceil x_1 \rfloor^{\frac{4+\alpha_2}{4}} \right]^{\frac{4+\alpha_3}{4+\alpha_2}} \right]^{\frac{4+\alpha_4}{4+\alpha_3}} \right]^0$$

  - Relay Polynomial Controllers: when $\alpha_\rho = \alpha_{\rho-1} = \cdots = \alpha_1 = \alpha \geq 0$

$$u_{2R} = -k_2 \mathrm{sign} \left[\lceil x_2 \rfloor^{2+\alpha} + \bar{k}_1 \lceil x_1 \rfloor^{\frac{2+\alpha}{2}} \right],$$

$$u_{3R} = -k_3 \mathrm{sign} \left[\lceil x_3 \rfloor^{3+\alpha} + \bar{k}_2 \lceil x_2 \rfloor^{\frac{3+\alpha}{2}} + \bar{k}_1 \lceil x_1 \rfloor^{\frac{3+\alpha}{3}} \right] \quad (1.11)$$

$$u_{4R} = -k_4 \mathrm{sign} \left[\lceil x_4 \rfloor^{4+\alpha} + \bar{k}_3 \lceil x_3 \rfloor^{\frac{4+\alpha}{2}} + \bar{k}_2 \lceil x_2 \rfloor^{\frac{4+\alpha}{3}} + \bar{k}_1 \lceil x_1 \rfloor^{\frac{4+\alpha}{4}} \right]$$

where for $\rho = 2$, $\bar{k}_1 = k_1^{2+\alpha}$; for $\rho = 3$, $\bar{k}_1 = k_2^{3+\alpha} k_1^{\frac{3+\alpha}{2}}$, $\bar{k}_2 = k_2^{3+\alpha}$; and for general ρ, $\bar{k}_i = \prod_{j=i}^{\rho-1} k_j^{\frac{\rho+\alpha}{\rho-j}}$, for $i = 1, \ldots, \rho - 1$. Relay Polynomial controllers are specially simple in its form.

- Quasi-Continuous Controllers
  - Nested Sliding Controllers: when some of the α_i are different. The parameters $\beta_i > 0$ are arbitrary

$$u_{2Q} = -k_2 \frac{\lceil x_2 \rfloor^{2+\alpha_2} + k_1^{2+\alpha_2} \lceil x_1 \rfloor^{\frac{2+\alpha_2}{2}}}{|x_2|^{2+\alpha_2} + \beta_1 |x_1|^{\frac{2+\alpha_2}{2}}},$$

$$u_{3Q} = -k_3 \frac{\lceil x_3 \rfloor^{3+\alpha_3} + k_2^{3+\alpha_3} \left[\lceil x_2 \rfloor^{\frac{3+\alpha_2}{2}} + k_1^{\frac{3+\alpha_2}{2}} \lceil x_1 \rfloor^{\frac{3+\alpha_2}{3}} \right]^{\frac{3+\alpha_3}{3+\alpha_2}}}{|x_3|^{3+\alpha_3} + \beta_2 |x_2|^{\frac{3+\alpha_3}{2}} + \beta_1 |x_1|^{\frac{3+\alpha_3}{3}}} \quad (1.12)$$

$$u_{4Q} = -k_4 \frac{\lceil x_4 \rfloor^{4+\alpha_4} + k_3^{4+\alpha_4} \left[\lceil x_3 \rfloor^{\frac{4+\alpha_3}{2}} + k_2^{\frac{4+\alpha_3}{2}} \left[\lceil x_2 \rfloor^{\frac{4+\alpha_2}{3}} + k_1^{\frac{4+\alpha_2}{3}} \lceil x_1 \rfloor^{\frac{4+\alpha_2}{4}} \right]^{\frac{4+\alpha_3}{4+\alpha_2}} \right]^{\frac{4+\alpha_4}{4+\alpha_3}}}{|x_4|^{4+\alpha_4} + \beta_3 |x_3|^{\frac{4+\alpha_4}{2}} + \beta_2 |x_2|^{\frac{4+\alpha_4}{3}} + \beta_1 |x_1|^{\frac{4+\alpha_4}{4}}}$$

– Relay Polynomial Controllers: when $\alpha_i = \alpha \geq 0$

$$
\begin{aligned}
u_{2QR} &= -k_2 \frac{\lceil x_2 \rfloor^{2+\alpha} + k_1^{2+\alpha} \lceil x_1 \rfloor^{\frac{2+\alpha}{2}}}{|x_2|^{2+\alpha} + \beta_1 |x_1|^{\frac{2+\alpha}{2}}}, \\
u_{3QR} &= -k_3 \frac{\lceil x_3 \rfloor^{3+\alpha} + \bar{k}_2 \lceil x_2 \rfloor^{\frac{3+\alpha}{2}} + \bar{k}_1 \lceil x_1 \rfloor^{\frac{3+\alpha}{3}}}{|x_3|^{3+\alpha} + \beta_2 |x_2|^{\frac{3+\alpha}{2}} + \beta_1 |x_1|^{\frac{3+\alpha}{3}}} \\
u_{4QR} &= -k_4 \frac{\lceil x_4 \rfloor^{4+\alpha} + \bar{k}_3 \lceil x_3 \rfloor^{\frac{4+\alpha}{2}} + \bar{k}_2 \lceil x_2 \rfloor^{\frac{4+\alpha}{3}} + \bar{k}_1 \lceil x_1 \rfloor^{\frac{4+\alpha}{4}}}{|x_4|^{4+\alpha} + \beta_3 |x_3|^{\frac{4+\alpha}{2}} + \beta_2 |x_2|^{\frac{4+\alpha}{3}} + \beta_1 |x_1|^{\frac{4+\alpha}{4}}}
\end{aligned}
\tag{1.13}
$$

All these controllers solve the problem posed in the previous Sect. 1.3.

Theorem 1.2 *For any $\rho \geq 2$ each controller of the families of Discontinuous or Quasi-continuous controllers in (1.8)–(1.9), with arbitrary parameters $0 \leq \alpha_1 \leq \cdots \leq \alpha_\rho$, is ρ-sliding homogeneous, and for k_ρ sufficiently large the ρth order sliding mode $x = 0$ is established in Finite-Time for the uncertain system (1.2) for properly chosen gains $k_1, \ldots, k_{\rho-1}$ and $\beta_1, \ldots, \beta_{\rho-1}$.*

In the case of measurement noise and/or perturbations we obtain (from the homogeneity [39, 40, 44]) the following accuracy properties (see Theorem 1.1).

Theorem 1.3 *Consider the uncertain plant (1.2) with any of the (state) feedback controllers (1.8) or (1.9) and suppose that conditions of Theorem 1.2 are satisfied. Suppose that the control is realized with a sampling interval τ. In this case the state x reaches after a finite time a neighborhood of the origin characterized by*

$$
|x_1(t)| \leq \delta_1 \tau^{\rho},\ |x_2(t)| \leq \delta_2 \tau^{\rho-1}, \ldots, |x_i(t)| \leq \delta_i \tau^{\rho-(i-1)},\ \left|x_\rho(t)\right| \leq \delta_\rho \tau,
$$

and x stays in this vicinity of zero for all future times. Here $\delta_1, \ldots, \delta_\rho > 0$ are some (positive) constants depending only on the chosen controller, the parameters (C, K_m, K_M, ρ) and the gains, but they are independent on τ and the initial conditions.

Using the CLF (1.5) we show that the convergence time is a bounded function of the initial states [8, Theorems 5.6, 5.7].

Proposition 1.1 *Controllers (1.8)–(1.9), in closed-loop with system (1.2), enforce the state trajectories, starting at initial state $x_0 = x(0) \in \mathbb{R}^n$, to reach $x = 0$ in a finite time smaller than*

$$
T(x_0) \leq m \eta_\rho V_\rho^{\frac{1}{m}}(x_0), \tag{1.14}
$$

where η_ρ is a function of the gains $(k_1, \ldots, k_\rho)$, K_m and C.

When the bound of the perturbation $[-C, C]$ is time-varying, it is possible to design the following variable-gain controller, with a slight variation of the proof of Theorem 1.2.

Theorem 1.4 *Consider that in* (1.2) $C = \bar{C} + \Theta(t, z)$*, where the function* $\Theta(t, z) \geq 0$ *is known. Then the discontinuous and quasi-continuous controllers* (1.8)–(1.9)*, with* k_ρ *replaced by the variable-gain* $\left(K(t, z) + k_\rho\right)$*, stabilize the origin* $x = 0$ *in Finite-Time if the gain* k_ρ *is chosen large enough and* $K_m K(t, z) \geq \Theta(t, z)$.

By making a linear change of variables $\zeta = Lx$, $L \geq 1$, it is easy to show that if the vector of gains $\mathbf{k} = (k_1, \ldots, k_\rho)$ is stabilizing, then so is the scaled vector of gains $\mathbf{k}_L = (L^{\frac{1}{\rho}} k_1, \ldots, L^{\frac{1}{\rho+1-i}} k_i, \ldots, L k_\rho)$ for any α_j. For the Relay Polynomial Controllers and the relative degree ρ, the gains $\bar{k}_i = \prod_{j=i}^{\rho-1} k_j^{\frac{\rho+\alpha}{\rho-j}}$, for $i = 1, \ldots, \rho - 1$ are scaled as $\bar{k}_i \to L^{\frac{(\rho-i)(\rho+\alpha)}{\rho-i+1}} \bar{k}_i$. Moreover, the convergence will be accelerated for $L > 1$, or the size of the allowable perturbation C will be incremented to LC. Note that the gains obtained by means of the LF can be very large for practical applications, so that a simulation-based gain design is eventually necessary (see [13]).

In next Sect. 1.5 it will be shown that the gains k_i can be calculated recursively as

$$k_1 > 0, \ldots, k_{i+1} > G_{i+1}(k_1, \ldots, k_i), \ldots, k_\rho > \frac{1}{K_m}\left(G_\rho\left(k_1, \ldots, k_{\rho-1}\right) + C\right),$$

where the functions G_i are obtained from the LF $V_c(x)$ and they depend on ρ, γ_i and α_i. We can parametrize the gains in terms of k_1 as

$$k_1 > 0, \ldots, k_i = \mu_i k_1^{\frac{\rho}{\rho-(i-1)}},\ k_\rho > \frac{1}{K_m}\left(\mu_\rho k_1^\rho + C\right), \tag{1.15}$$

for some positive constants μ_i depending on ρ, γ_i and α_i. Some values, calculated numerically for u_D in (1.11), for $\alpha = 0$, are: $\rho = 2$, $\mu_2 = 1.62$; $\rho = 3$, $(\mu_2 = 1.5,\ \mu_3 = 3.25)$; $\rho = 4$, $(\mu_2 = 2,\ \mu_3 = 8.45,\ \mu_4 = 30)$. We note that this parametrization can be used for all controllers, except the value of k_ρ, which is different for the discontinuous and the quasi-continuous controllers. The values can also be used with the variable gain controller.

Remark 1.1 We note that using the family of CLFs (1.5) we obtain a large family of HOSM controllers, related to the ones proposed by A. Levant in his works. However, not all Levant's controllers have been provided with a Lyapunov function. For example, [19] derive Polynomial controllers for arbitrary $\alpha > -\rho$, while the Lyapunov functions proposed here (and also in [19]) are only valid for $\alpha \geq 0$. The construction of Lyapunov functions for the controllers for the values of α in the interval $-\rho < \alpha < 0$ is an open problem.

Remark 1.2 It is easy to see that controllers (1.8) and (1.9) can be modified without changing their properties: suppose that $\varsigma_\rho(x)$ is a continuous $\mathbf{r}$-homogeneous function of degree $\rho + \alpha_\rho$ such that (i) $\left\{x \in \mathbb{R}^\rho \mid \varsigma_\rho(x) = 0\right\} = \left\{x \in \mathbb{R}^\rho \mid \sigma_\rho(x) = 0\right\}$, i.e. σ_ρ and ς_ρ vanish at the same points, and (ii) $\varsigma_\rho(x)\sigma_\rho(x) \geq 0$. In this case the controllers

$$u_D = -k_\rho \left\lceil \varsigma_\rho(x) \right\rfloor^0 ,$$
$$u_Q = -k_\rho \frac{\varsigma_\rho(x)}{M(x)} ,$$

have the same properties as (1.8) and (1.9), respectively. For example, instead of the controller u_{3D} in (1.10) we can also implement the controller

$$u_{3D} = -k_3 \left\lceil \left\lceil \lceil x_3 \rfloor^{3+\alpha_2} + k_2^{3+\alpha_2} \lceil x_2 \rfloor^{\frac{3+\alpha_2}{2}} \right\rfloor^{\frac{3+\alpha_3}{3+\alpha_2}} + k_1^{\frac{3+\alpha_3}{2}} k_2^{3+\alpha_3} \lceil x_1 \rfloor^{\frac{3+\alpha_3}{3}} \right\rfloor^0 .$$

1.5 The Lyapunov Function for the HOSM Controllers

This section can be skipped in a fast reading. We show that the continuously differentiable function $V_\rho(x)$ (1.5) is an **r**-homogeneous Lyapunov Function of degree m for the uncertain system (1.2) for all $\rho \geq 2$, all $\gamma_j > 0$ and sufficiently large values of $k_j > 0$, $j = 1, 2, \ldots, \rho - 1$.

1.5.1 Proof by the Lyapunov Approach

In this subsection we establish the relationship between the proposed family of controllers and the Control Lyapunov functions. First we present some preliminary results.

We define recursively the auxiliary variables

$$s_1 = x_1, \ldots, s_i = x_i - \nu_{i-1}(\bar{x}_{i-1}) ,$$

$$s_{1,d} = \lceil x_1 \rfloor^{\frac{m-r_1}{r_1}}, \ldots, s_{i,d} = \lceil x_i \rfloor^{\frac{m-r_i}{r_i}} - \lceil \nu_{i-1}(\bar{x}_{i-1}) \rfloor^{\frac{m-r_i}{r_i}} .$$

Lemma 1.2 *For $\alpha > 1$ and $\beta > 0$ consider the function of the two real variables $x, y \in \mathbb{R}$*

$$F(x, y) = \frac{1}{\alpha} |x|^\alpha - x \lceil y \rfloor^\beta + (1 - \tfrac{1}{\alpha}) |y|^{\beta \frac{\alpha}{\alpha - 1}} . \tag{1.16}$$

Then $F(x, y) \geq 0$ and $F(x, y) = 0$ if and only if $\lceil x \rfloor^\alpha = \lceil y \rfloor^{\beta \frac{\alpha}{\alpha-1}}$.

Proof The conclusion follows immediately from Young's inequality (see Lemma 1.4), since it implies

$$x \lceil y \rfloor^\beta \leq \frac{1}{\alpha} |x|^\alpha + \left(1 - \frac{1}{\alpha}\right) |y|^{\beta \frac{\alpha}{\alpha-1}} .$$

□

From Lemma 1.2 it follows that $W_i(\bar{x}_i) \geq 0$ and $W_i(\bar{x}_i) = 0$ iff $x_i = \nu_{i-1}(\bar{x}_{i-1})$, so that $V_i(\bar{x}_i)$ is positive definite for any $\gamma_{i-1} > 0$. The following relations will be used in the sequel: for $1 \leq j \leq i-1$

$$\frac{\partial W_i(\bar{x}_i)}{\partial x_j} = -\frac{m - r_i}{\rho + \alpha_{i-1}} k_{i-1}^{\frac{\rho+\alpha_{i-1}}{r_i}} |\nu_{i-1}|^{\frac{m-r_i-\rho-\alpha_{i-1}}{r_i}} s_i \frac{\partial \sigma_{i-1}(\bar{x}_{i-1})}{\partial x_j}, \tag{1.17}$$

$$\frac{\partial W_i(\bar{x}_i)}{\partial x_i} = \lceil x_i \rfloor^{\frac{m-r_i}{r_i}} - \lceil \nu_{i-1}(\bar{x}_{i-1}) \rfloor^{\frac{m-r_i}{r_i}} = s_{i,d}. \tag{1.18}$$

Notice also that $s_i = 0 \Leftrightarrow \sigma_i = 0 \Leftrightarrow s_{i,d} = 0$, and that they have the same sign, i.e. $s_i \sigma_i > 0$, $s_i s_{i,d} > 0$ and $\sigma_i s_{i,d} > 0$ when $s_i \neq 0,\ \sigma_i \neq 0,\ s_{i,d} \neq 0$.

For $i = 1, \ldots, \rho - 1$ we introduce also the functions

$$Z_1(\bar{x}_1) = \nu_1(\bar{x}_1) \frac{dV_1(x_1)}{dx_1},\ Z_i(\bar{x}_i) \triangleq \sum_{j=1}^{i-1} x_{j+1} \frac{\partial V_i(\bar{x}_i)}{\partial x_j} + \nu_i(\bar{x}_i) \frac{\partial V_i(\bar{x}_i)}{\partial x_i}. \tag{1.19}$$

Using (1.5) in (1.19), and with (1.17), (1.18) we obtain

$$\begin{aligned} Z_i(\bar{x}_i) = & \gamma_{i-1} \left(\sum_{j=1}^{i-2} x_{j+1} \frac{\partial V_{i-1}(\bar{x}_{i-1})}{\partial x_j} + x_i \frac{\partial V_{i-1}(\bar{x}_{i-1})}{\partial x_{i-1}} \right) \\ & - \frac{m - r_i}{\rho + \alpha_{i-1}} k_{i-1}^{\frac{\rho+\alpha_{i-1}}{r_i}} |\nu_{i-1}|^{\frac{m-r_i-\rho-\alpha_{i-1}}{r_i}} s_i \sum_{j=1}^{i-1} x_{j+1} \frac{\partial \sigma_{i-1}(\bar{x}_{i-1})}{\partial x_j} + s_{i,d} \nu_i. \end{aligned}$$

Using $x_i = \nu_{i-1} + s_i$ in the first line we can obtain the recursive expression

$$\begin{aligned} Z_i(\bar{x}_i) &= \gamma_{i-1} Z_{i-1}(\bar{x}_{i-1}) + s_i \Psi_{i-1}(\bar{x}_i) + s_{i,d} \nu_i \\ \Psi_{i-1} &\triangleq \gamma_{i-1} s_{i-1,d} - \tfrac{m-r_i}{\rho+\alpha_{i-1}} k_{i-1}^{\frac{\rho+\alpha_{i-1}}{r_i}} |\nu_{i-1}|^{\frac{m-r_i-\rho-\alpha_{i-1}}{r_i}} \sum_{j=1}^{i-1} x_{j+1} \frac{\partial \sigma_{i-1}(\bar{x}_{i-1})}{\partial x_j} \end{aligned} \tag{1.20}$$

Note that if in (1.19) we set $\frac{\partial V_i(\bar{x}_i)}{\partial x_i} = s_{i-1,d} = s_i = 0$ it follows from (1.20) that

$$\sum_{j=1}^{i-1} x_{j+1} \frac{\partial V_i(\bar{x}_i)}{\partial x_j} = \gamma_{i-1} Z_{i-1}(\bar{x}_{i-1}). \tag{1.21}$$

The Main Argument

The proof can be divided in two parts. We take the derivative of $V_\rho(x)$ along the trajectories of (1.2)

$$\dot{V}_{\rho}(x) \in \sum_{j=1}^{\rho-1} \frac{\partial V_{\rho}(x)}{\partial x_j} x_{j+1} + \frac{\partial V_{\rho}(x)}{\partial x_{\rho}} \left([-C,\ C] + [K_m,\ K_M]u\right) ,$$

with the controller (1.8) or (1.9). Note that

$$\frac{\partial V_{\rho}(x)}{\partial x_{\rho}} = \frac{\partial W_{\rho}(x)}{\partial x_{\rho}} = s_{\rho,d}(x) = \lceil x_{\rho} \rfloor^{\frac{m-r_{\rho}}{r_{\rho}}} - \lceil \nu_{\rho-1}(\bar{x}_{\rho-1}) \rfloor^{\frac{m-r_{\rho}}{r_{\rho}}}$$

and

$$\sigma_{\rho}(x) = \lceil x_{\rho} \rfloor^{\frac{\rho+\alpha_{\rho}}{r_{\rho}}} - \lceil \nu_{\rho-1} \rfloor^{\frac{\rho+\alpha_{\rho}}{r_{\rho}}} = \lceil x_{\rho} \rfloor^{\frac{\rho+\alpha_{\rho}}{r_{\rho}}} + k_{\rho-1}^{\frac{\rho+\alpha_{\rho}}{r_{\rho}}} \lceil \sigma_{\rho-1} \rfloor^{\frac{\rho+\alpha_{\rho}}{\rho+\alpha_{\rho-1}}} .$$

Moreover, $s_{\rho,d}(x)\,\sigma_{\rho}(x) \geq 0$ and it is zero only if $s_{\rho,d}(x) = \sigma_{\rho}(x) = 0$.

We consider here only the Discontinuous controller (1.8), the proof for the Quasi-Continuous one follows the same path. For (1.8) the multivalued function $\psi(u) = [-C,\ C] - k_{\rho}[K_m,\ K_M]\,u$, when evaluated at (1.8), i.e.

$$\psi(\sigma_{\rho}(x)) = [-C,\ C] - k_{\rho}[K_m,\ K_M] \lceil \sigma_{\rho}(x) \rfloor^{0} ,$$

can be represented as

$$\psi(\sigma_{\rho}(x)) = \begin{cases} -\left[(C + k_{\rho}K_m),\ (k_{\rho}K_M - C)\right] & \text{if } \sigma_{\rho}(x) > 0 \\ \left[-(C + k_{\rho}K_M),\ (C + k_{\rho}K_M)\right] & \text{if } \sigma_{\rho}(x) = 0 \\ \left[(k_{\rho}K_m - C),\ (C + k_{\rho}K_M)\right] & \text{if } \sigma_{\rho}(x) < 0 \end{cases}$$

If we assume that $\frac{C}{K_m} < k_{\rho}$ then we conclude that

$$\psi(\sigma_{\rho}(x))\,\sigma_{\rho}(x) \leq 0\,,\ \text{and}\ \psi(\sigma_{\rho}(x))\,s_{\rho,d}(x) \leq 0\,,$$

and they are zero only if $\sigma_{\rho}(x) = 0$.

Using these results we conclude that

$$\dot{V}_{\rho}(x) \leq \sum_{j=1}^{\rho-1} \frac{\partial V_{\rho}(x)}{\partial x_j} x_{j+1} - k_{\rho}\left(K_m - \frac{C}{k_{\rho}}\right) \left| s_{\rho,d}(x) \right| .$$

If we assume that

$$\forall x \in \left\{x \in \mathbb{R}^{\rho} | s_{\rho,d}(x) = 0\right\} = \left\{x \in \mathbb{R}^{\rho} | \sigma_{\rho}(x) = 0\right\} \Rightarrow \sum_{j=1}^{\rho-1} \frac{\partial V_{\rho}(x)}{\partial x_j} x_{j+1} < 0 \tag{1.22}$$