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Alexander Soifer *Editor*

Competitions for Young Mathematicians

Perspectives from Five Continents

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Alexander Soifer
Editor

Competitions for Young Mathematicians

Perspectives from Five Continents

With the Foreword by Gabriele Kaiser

Springer

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This book is dedicated to all those people around the world
Who are passing baton to next generations of mathematicians

Foreword

Mathematical competitions are a chance for mathematically talented young scholars to experience mathematics as a research-oriented discipline. These competitions offer the chance to get insight into the beauty of mathematical structures at a high level, which many of these young mathematicians usually will not experience at home. Furthermore, these competitions allow to meet other talented young mathematicians, exchange their ideas with them and experience that they are not singular and isolated youngsters, but part of an important community.

Despite this high importance of mathematical competitions, either as mathematical Olympiad or as mathematical tournament of towns or other kinds of mathematical competitions, there exists hardly any scientific research about mathematical competitions. This is surprising, because these mathematical competitions have a long tradition and a high influence on the promotion of young talented mathematicians.

At the occasion of the 13th International Congress on Mathematical Education (ICME-13) a Topic Study Group on Mathematics Competitions took place, at which famous researchers working in this field met and exchanged about the state-of-the-art in this field. This intensive work together with papers from related groups forms the basis of this book.

The book provides an excellent overview about the current discussion, topical themes and experiences with mathematical competitions. It starts with reflections on goals of mathematics education, problems coming from geometry or combinatorics being used in mathematical competitions. The next parts reflect on the role of competitions in the classroom, this theme is hardly researched so far. Then two examples of mathematical competitions are analyzed. The last two parts focus on the present state of mathematical competitions and its future and a bridge between competitions and 'real' mathematics.

To summarize, this book is more than overdue and reflects from an academic perspective on the potential of mathematical competitions for mathematics education in general.

I wish to congratulate the editor—Alexander Soifer—and the contributors to this timely and excellent book.

Hamburg, Germany

Gabriele Kaiser
Convenor of the 13th International Congress
on Mathematical Education, University of Hamburg

Preface

The role and usefulness of competitions in mathematics instruction has been debated for decades. If memory holds, I attended a deep and entertaining debate on this topic between a distinguished mathematician Peter John Hilton and a renowned math educator Gilah C. Leder at ICME-6, held in 1988 in Budapest. As this volume demonstrates, competitions problems can be used to enrich classroom instruction, to offer our students an exciting pastime, to raise interest in mathematics, and to enable students to commence their mathematical research. If not for Moscow State University Olympiads and a mathematical circle conducted by Nikolai Konstantinov (one of the authors in this volume!), I would have become a classical pianist and composer and not a mathematician. (By no means am I suggesting here that mathematics is better than music—they both belong to the Pantheon of the Arts.)

I am duty bound to add one warning. If a student does consistently well in mathematical Olympiads, s(he) clearly has a talent, and with a good measure of interest and hard work will go far. However, no discouraging conclusion could be made about a student, who has not sparkled in the Olympiads. Young people develop at diverse speeds. Moreover, mathematics competitions inevitably have an element of sports, the necessity to perform under pressure and within a limited time. High speed of thinking is attractive, but it is not an essential property for a future successful researcher.

This book includes plenary talks and some of the best presentations made in the Topic Study Group 30: Mathematics Competitions of the International Congress on Mathematical Education (ICME-13) in Hamburg, and some of the best presentations from related groups, dedicated to work with gifted students and mathematical enrichment. Each of the chapters, on request of this editor, includes not only original ideas of pedagogy and state-of-the-art methods of mathematical instruction, but also original problems and their beautiful solutions. I believe that this volume will be a valuable addition to the mathematics literature for secondary teachers and university professors around the world, and their gifted students of all levels, from secondary to graduate students, seeking problems to start their research careers.

The authors of this book comprise a group that impresses me enormously. It includes seven laureates of the Paul Erdős Award and one of the David Hilbert Award presented by the World Federation of National Mathematics Competitions (WFNMC); three past or present Presidents of WFNMC; five past or present WFNMC's Vice Presidents; three WFNMC's Secretaries; laureates of numerous other awards, leaders of and contributors to ICMI studies; authors of many books and countless articles, organizers of the International Mathematical Olympiad (IMO). In fact, in 1994 and 2016, K. P. Shum was the Organizer of two IMO's held in Hong Kong; while in 2013 Maria Falk de Losada served as the President of the International Jury at the Colombian IMO. The authors include many leaders and deputy leaders of national teams IMO teams, coordinators of IMO, organizers of numerous national and international competitions, conferences and congresses, etc.

Each of the 14 chapters addresses many issues and contributes to a multitude of directions, which makes a partition of the material into parts nearly impossible. I attempted to identify the main direction of each chapter and thus help the reader by partitioning the book into seven parts. As you can see, Francisco Bellot-Rosado (Spain) and Kar-Ping Shum (P.R. China) present problems of geometry; Kiril Bankov (Bulgaria), and Luis F. Cáceres-Duque, Jose H. Nieto-Said, and Rafael Sánchez-Lamoneda (Puerto Rico) share combinatorial problems. Role of competitions for a classroom is described by Robert Geretschläger (Austria); Ingrid Semanišinová, Matúš Harminc, and Martina Jesenská (Slovakia); and Iliana Tsvetkova (Bulgaria). Nikolai Konstantinov and Sergei Dorichenko (Russia), describe their famous International Mathematical Tournament of Towns; V.M. Sholapurkar (India) presents a relatively recent competition for college students. Romas Kasuba (Lithuania) shares his lifetime experiences with competitions; while Peter Taylor (Australia) classifies problems of mathematics competitions. Maria Falk De Losada (Colombia) collects valuable observations of the influence of mathematics competitions on their contestants, destined to become world's leading mathematical researchers. Alexander Soifer (USA) opens the book with his view of goals and means of mathematics instruction and closes the book with examples of bridges between problems of mathematical Olympiads and research problems of 'real' mathematics.

It was a delight to organize and run the Topic Study Group jointly with Maria Falk de Losada, thank you, Maria! My gratitude goes to my referees, encompassing four continents, who helped the authors to improve their

chapters in a significant way. I thank all the officials and volunteers of ICME-13 in Hamburg, who allowed us all a pleasure of sharing knowledge and experiences during this Olympics-like forum of nearly 4,000 professionals from 109 countries. My special thanks go to the Convenor and the Chair of the International Program Committee of the ICME-13 Prof. Dr. Gabriele Kaiser for creating the Congress and arranging this splendid opportunity for my group of 18 authors from five continents to unite in a truly Olympic spirit and produce this volume, and to Springer for making it possible for us to preserve the wonderful memories of the Hamburg Congress in the form of this book.

On behalf of all the authors of this book, I wish you, our reader, to get much pleasure of mathematical kind from this book and many other books written by these 18 authors.

Colorado Springs, USA
January 2017

Alexander Soifer

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Part I
Goals of Mathematics Instruction

Chapter 1
Goals of Mathematics Instruction: Seven Thoughts and Seven Illustrations of Means

Alexander Soifer

Abstract The goal of this chapter is to present what the author sees as the state-of-the-art approach to mathematics instruction, and the state-of-the-art use of mathematical Olympiads in bringing instruction closer to 'real' mathematics and identifying young talents. One of the principle goals of mathematics instruction ought to be showing in a classroom what mathematics is and what mathematicians do. This cannot be achieved by teaching but rather by creating an environment in which students learn mathematics by doing it. As in 'real' mathematics, this ought to be done by solving problems that require not just plugging numbers into memorized formulas and one-step deductive reasoning, but also by experimenting, constructing examples, and utilizing synthesis in a single problem of ideas from various branches of mathematics, built on high moral foundations. The author's eight recent Springer books present fragments of 'live' mathematics, and illustrations of these ideas. The chapter also describes the role of mathematical olympiads in instruction and includes some problems used at the Colorado Mathematical Olympiad over the past 34 years.

This essay is an expanded version of the Plenary Talk in the *Topic Study Group 30: Mathematics Competitions* at the 13th International Congress on Mathematical Education, Hamburg, Germany, July 2016. Prof. Dr. Gabriele Kaiser was the Convener of this very successful Congress. The early version appeared in the journal of the World Federation of National Mathematics Competitions 29(1), 2016, 7–30.

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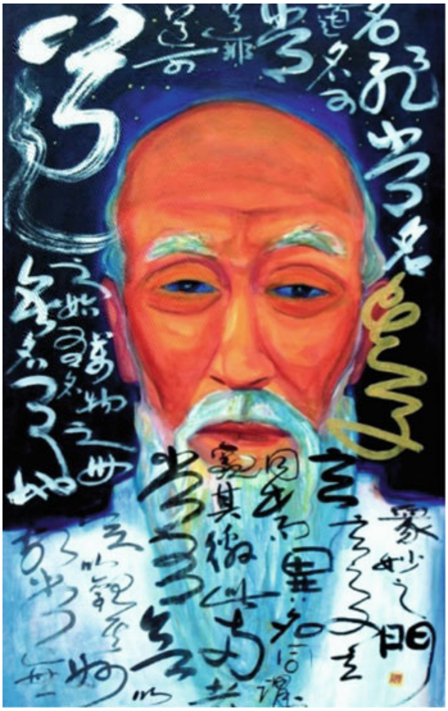
1.1 Part I: Seven Thoughts on Mathematics Instruction

Give a man a fish, and you will feed him for a day.

Teach a man how to fish, and you will feed him for a lifetime.

– 老子 (Lǎozǐ, VI century BC)



1. The Purpose of Life Implies the Purpose of Instruction

Before we address the purpose of mathematics instruction, it makes sense to ask ourselves, what is the purpose of life itself? It seems to me that the purpose of life is to discover and express ourselves, and in so doing contribute to high culture of our planet. The ultimate purpose of instruction is therefore to aid our students in their quest for self-discovery and self-expression.

2. A Typical Instruction: Dishing out a Collection of Facts a la "Give a Man a Fish"

Instruction is often reduced to memorization of a certain collection of facts: dates in history, theorems in mathematics, etc. While memorization and knowledge are of value, they seem to be overestimated in instruction. I agree with the great Chinese Sage Lǎozǐ: giving a man a fish will not solve man's problem of survival.

3. Lǎozǐ and a Skill Approach to Life: "Teach a Man How to Fish"

Lǎozǐ proposes to teach a man fishing as a method of solving the problem of survival. This does go further than giving a man a fish. However, is it good enough in today's world?

4. Beyond Lǎozǐ: Enable a Man to Learn How to Solve Problems

Not every education is as good an investment as another. We ought to go beyond Lǎozǐ and his universally celebrated lines. Is teaching skills good enough? Not quite, dear Sage, not in today's rapidly changing world. What if there is no more fish? What if the pond has dried out while your man has only one skill, fishing?

A problem solver will not die if the fish disappears in a pond—he'll learn to hunt, grow crop, solve whatever problems life puts in his way. And so, we will go a long way by putting emphasis not on training skills but on creating environment for developing problem solving abilities and attitudes. This is the state-of-the-art. The proverb for today's world ought to be:

Give a man skills, and you will feed him in the short run.

Let a man learn solving problems, and you will feed him for a lifetime.

5. Mathematics and Life

Every day we confront and solve a myriad of problem. Life *is* about solving problems. And mistakes in solving life's problems could be quite costly: a bridge could collapse, electrical grid could get overloaded, traffic could get to a halt, etc. This is where mathematics comes in handy. Mathematics allows us to learn how to think creatively, how to solve problems. And once our student masters problem solving in mathematics, s(he) will be better prepared to confront problems in any human endeavor.

6. Are the Two Popular Approaches to Mathematics Instruction Good Enough?

Today's discussions of mathematical instruction seem to be reduced to two competing approaches, "Embrace the Technology" versus "Back to the Basics."

"Back to the Basics" is not the best solution, for it emphasizes mind numbing drill, and treats students as robots, who need to be pre-programmed with a set of skills. In the newer "Embrace the Technology" approach, I support taking a teacher off the lectern and letting students work on their own. This approach too more often than not treats students like robots, and pre-programs them with skills of today. However, technology nowadays changes rapidly, as do the societal demands for particular skills.

Providing public education is not only an ethical thing to do—it is a profitable investment. Are there many jobs today for computer-illiterate persons? And yet just one generation ago, computers were a monopoly of researchers, and one generation before that did not exist at all. And so, we will go a long way by putting emphasis not on training skills but on creating atmosphere for developing problem solving abilities and attitudes.

Observe, one *cannot teach* mathematics, or anything else for that matter. State-of-the-art in mathematics instruction is about creating an atmosphere where students can learn mathematics by doing it, with a gentle guidance of a teacher.

7. The True Goal of Mathematics Instruction is to Demonstrate What Mathematics Is and What Mathematicians Do

Standardized three-letter tests, such as SAT, ACT, GRE, KGB, CIA (well, the latter two triples are from a different opera:-) can only inform us how well a student does on these tests. Is this the goal of instruction? We ought to abandon standardized multiple choice testing of skills. There are more important things to assess. Over the past 34 years, *The Colorado Mathematical Olympiad* has been offering middle and high school students 5 original problems of increasing difficulty and 4 hours to think, to invent, and to solve. We "test" predominantly not knowledge, not skills, but creativity and originality of thought (Soifer 2011–2; Soifer 2017).

Is the goal "teaching to the test," as the past USA President George W. Bush believed? Not really. We all agree that problem solving is the means of instruction. However, what is *problem solving*? A typical secondary school problem asks to "find the hypotenuse of a right triangle, whose legs are 3 and 4, by using Pythagoras Theorem." No, not any more, you would reply. Nowadays, at the Age of Technology, a typical secondary school problem asks to "find the hypotenuse of a right triangle, whose legs are 3.1 and 4.2, by using Pythagoras Theorem and your smartphone." Would you call it a progress?

More generally, a secondary school problem has the structure $\mathbf{A} \Rightarrow \mathbf{B}$, i.e., given **A** prove **B** by using theorem **C**. In real life, no one gives a research mathematician a **B**; it is discovered by intuition and is based on experimentation. And of course, no one knows a **C** since nobody solved the problem: a research mathematician is a pioneer, moving along an untraveled path!

And so, we ought to bring our secondary and college mathematics, which often looks so superficial, as close as possible to the 'real' mathematics. We ought to let our students experiment in our classroom-laboratory. We ought to let them develop intuition and use it to come up with a conjecture **B**. And we ought to let our students find those tools **C** that do the job of deductive proving the conjecture **B**. In my opinion, the true goal of mathematics instruction is to demonstrate in the classroom *what mathematics is*, and *what mathematicians do.*

8. What Can Mathematical Olympiads Bring to Mathematics Instruction?

Let us first of all define the term. A *mathematical olympiad* is a competition where contestants are required to write essay-type complete solutions of the problems. Number of problems offered to Olympians is relatively small, usually between 4 to 6, and the time allowed is relatively long, usually from 4 to 9 hours. This does not completely eliminate time as a factor affecting performance, but substantially reduces it, especially compared to multiple choice or answer-only competitions with their speed-guessing as the main virtue. I often see best Olympians continuing to think about difficult problems after the Olympiad ends. In fact, I know some of them, who have been thinking about a Colorado Mathematical Olympiad problem and its research generalizations for many years. This process and the Olympiad influence may last a lifetime. While I see value in quick-type mathematical competitions and its sporty attraction for television broadcasting, I personally do not think they faithfully represent what mathematics is and what mathematicians do.

Olympiads allow us to introduce secondary students to topics, ideas, and methods of 'real' mathematics in the context and terminology of secondary mathematics, in the form that is digestible by them. Problems of mathematical Olympiads—as not much else—demonstrate beauty and elegance of mathematics. At the age of 14, I switched from writing and performing piano music to mathematics due exclusively to *The Moscow Mathematical Olympiad*. In March 1989 in Colorado Springs, Paul Erdős told me that "the Olympiads create a new enthusiasm toward mathematics, and in this sense are very valuable."

At *The Colorado Mathematical Olympiad*, we have been often asked a natural question: how does one create a mathematical Olympiad? This and other related questions are clarified by the University of Colorado, which produced the film "*Thirtieth Colorado Mathematical Olympiad—30 Years of Excellence*" that can be found on the Olympiad's homepage http://olympiad.uccs.edu/.

9. The Moral Foundation Is Critical

There is an opinion shared by many of my colleagues that all that matters is mathematics, *Mathematik über Alles*, if you will, above all moral concerns. In my opinion, there is no good science or good art unless it is built on the foundation of high ethical principles. Luitzen Egbertus Jan Brouwer, a great Dutch mathematician and philosopher, wrote in his 1929 letter: "It is my opinion that the tiniest moral matter is more important than all of science, and that one can only maintain the moral quality of the world by standing up to any immoral project."

We have seen in history time and again how evil the usage of science could be if it is not built on high moral foundation. Atrocities of Nazi Germany alone provide countless examples of science, technology and even art used for ill deeds. My book (Soifer 2015) is dedicated to moral dilemmas of a scholar in the Third Reich and in the world of today. Lessons of history ought to enter our classrooms and give moral guidance to our students today. I value education, however, I must admit that

Fine education does not guarantee high culture,

And high culture does not guarantee humanity.

In order for creative work to be good, it must also serve the good. It ought to be humane. It has to be grounded in morality, empathy, compassion, and kindness. The Great Russian poet Alexander Pushkin (1799–1837) beautifully wrote about it. Let me translate his lines for you:[1]

And people will be pleased with me for years to come,
For I awakened kindness with my lyre,
For in my cruel age I Freedom praised and sang
And urged I mercy for the fallen people.

And so we ought to pass to our students the baton of mercy and humanity, so that our students by their creative work contribute to the high culture of our small endangered planet.

1.2 Part II: Seven Illustration of Means

Alright, but what kind of problems should we offer our students? What approaches should we present in our classrooms? Permit me to illustrate seven essential components of the state-of-the-art classroom.

1. Experiment in Mathematics

First of all, we ought to set up a *mathematical laboratory*, where students conduct mathematical experiments, develop inductive reasoning and an insight needed to create conjectures. Some illustrations of it can be found in (Soifer 2010–1). For example, a short experiment allows us to conjecture a formula for the sum of cubes of consecutive integers:

$$\begin{aligned} 1^3 &= 1^2 \\ 1^3 + 2^3 &= 3^2 \\ 1^3 + 2^3 + 3^3 &= 6^2 \\ 1^3 + 2^3 + 3^3 + 4^3 &= 10^2 \end{aligned}$$

We observe that the sums of consecutive cubes are perfect squares. But squares of what numbers? If you are not able to develop a conjecture yet, continue to experiment: $1^3 + 2^3 + 3^3 + 4^3 + 5^3 = 15^2$. You will soon

[1]И долго буду тем любезен я народу,
Что чувства добрые я лирой пробуждал,
Что в мой жестокий век восславил я Свободу
И милость к падшим призывал.

notice that $1^3 + 2^3 + 3^3 + 4^3 + 5^3 = (1 + 2 + 3 + 4 + 5)^2$. This kind of equality holds for all the values in our experiment, and the conjecture is ready:

$$1^3 + 2^3 + \cdots + n^3 = (1 + 2 + \cdots + n)^2.$$

We can now prove, for example, by mathematical induction, that both the left side and the right side of the conjectured equality is equal to $\left(\frac{n(n+1)}{2}\right)^2$. ■

2. Construction of Examples in Mathematics

Construction of counterexamples is almost non-existent in secondary education and even university, whereas counterexamples play a major role in mathematics, amounting to circa 50% of its results. In fact, the Great Russian mathematician Israel M. Gelfand once said, "Theories come and go; examples live forever."

You would agree that practically the entire school mathematics consists of analytical proofs. In order to bring instruction closer to the 'real' mathematics we ought to include in education construction of examples and counterexamples. Let me share one example, where a construction solves the problem (Soifer 2011–2).

Positive² (18th Colorado Mathematical Olympiad, Soifer 2001). Is there a way to fill a 2001×2001 square table T with pluses and minuses, one sign per cell of T, such that no series of interchanging all signs in any 1000×1000 or 1001×1001 square of the table can fill T with all pluses?

Solution. Having created this problem and its solution for the 2001 Colorado Mathematical Olympiad, I felt that another solution was possible using an invariant, but failed to find it. Two days after the Olympiad, on April 22, 2001, the past double-winner of the Olympiad Matthew Kahle, now a Professor at Ohio State University, found the solution that eluded me. It is concise and beautiful.

Define (see Fig. 1.1) Φ = {the set of all cells of T, except those in the middle row}. Observe that no matter where a 1000×1000 square S is placed in the table T, it intersects Φ in an even number of cells, because there are 1000 equal columns in S. Observe also that no matter where a 1001×1001 square S' is placed in T, it also intersects Φ in an even

Fig. 1.1 .

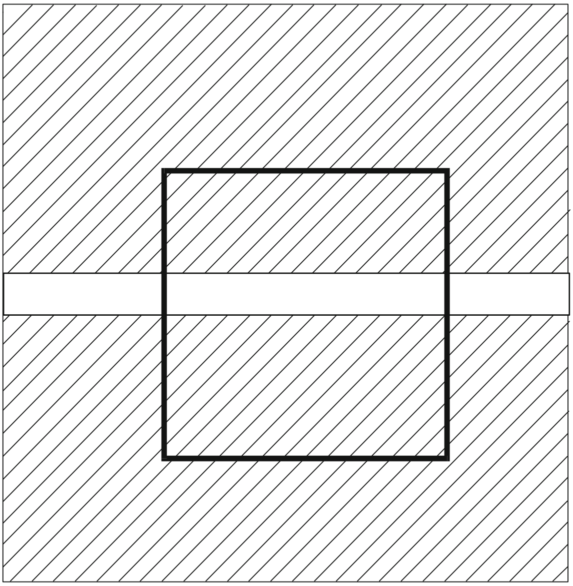
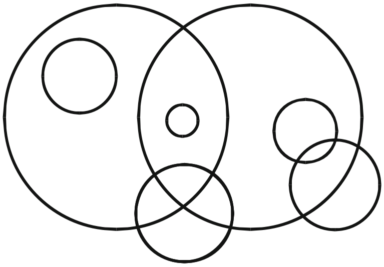


Fig. 1.2 .



number of unit squares, because there are 1000 equal rows in *S'* (one row is always missing, since the middle row is omitted in *S*.)

Now we can easily create the required assignment of signs in *T* that cannot be converted into all pluses. Let Φ have any assignment with an *odd* number of + signs, and the missing in Φ middle row be assigned signs in any way. No series of operations can change the parity of the number of pluses in Φ, and thus no series of allowed operations can create all pluses in Φ. ■

3. Utilizing Analogy

A sense of analogy could be a powerful tool. Here is one example from (Soifer 2009–2).

Problem 2 Prove that a map formed in the plane by finitely many circles can be 2-colored (Fig. 1.2).

Fig. 1.3 .

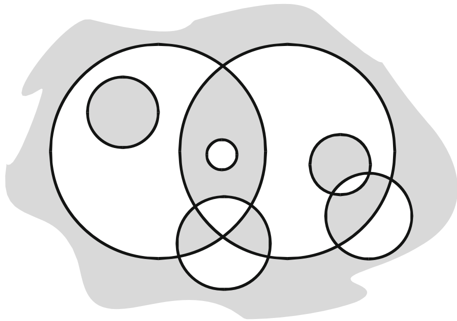
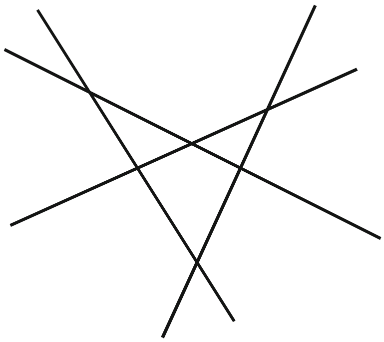


Fig. 1.4 .



Proof We partition regions of the map into two classes (Fig. 1.3): those contained in an even number of circles (color them gray), and those contained in an odd number of circles (leave them white). Clearly, neighboring regions got different colors because when we travel across their boundary line, the parity changes. ■

I am sure you realize that the shape of a circle is of no consequence. We can replace circles in problem 2 by simple closed curves. However, can we replace simple closed curves by straight lines?

Problem 3 Prove that a map formed in the plane by finitely many straight lines is 2-colorable (Fig. 1.4).

An inductive proof is well known, but, as is usually the case with proofs by mathematical induction, it does not provide an insight. Decades ago I found a 'one-line' proof that takes advantage of similarity between simple closed curves and straight lines.

Proof Attach to each line a vector perpendicular to it (Fig. 1.5). Call the half-plane *inside* if contains the vector, and *outside* otherwise. Repeat the proof of problem 2 word-by-word to complete the proof (Fig. 1.6). ■

Fig. 1.5 .

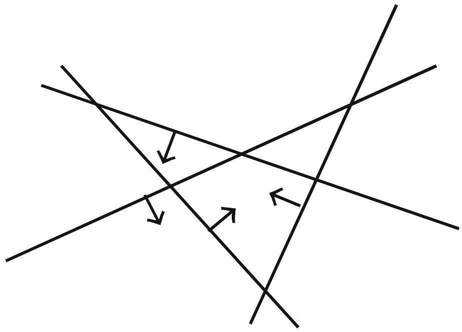
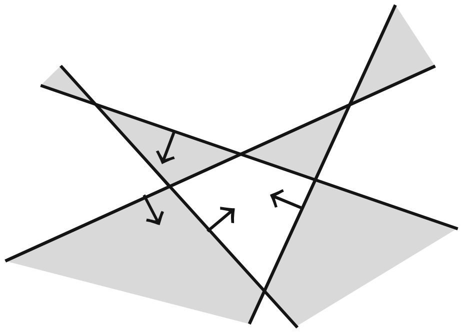


Fig. 1.6 .



4. Method and Anti-method

Tiling with Dominoes. **(Method)**. Can a chessboard with two diagonally opposite squares missing, be tiled by dominoes (Fig. 1.7)?

Solution. Color the board in a chessboard fashion (Fig. 1.8). No matter where a domino is placed on the board, vertically or horizontally, it would cover one black and one white square. Thus, it is necessary for tileability to have equal numbers of black and white squares in the board—but they are not equal in our truncated board. Therefore, the required tiling does not exist. ■

It is impressive and unforgettable for a student to see for the first time how coloring can solve a mathematical problem. However, I noticed that once a student learns a coloring idea, s(he) always resorts to it when a chessboard and dominoes are present in the problem. This is why I created the following 'Anti-Method' Problem and used it in the Colorado Mathematical Olympiad (Soifer 2011–2).

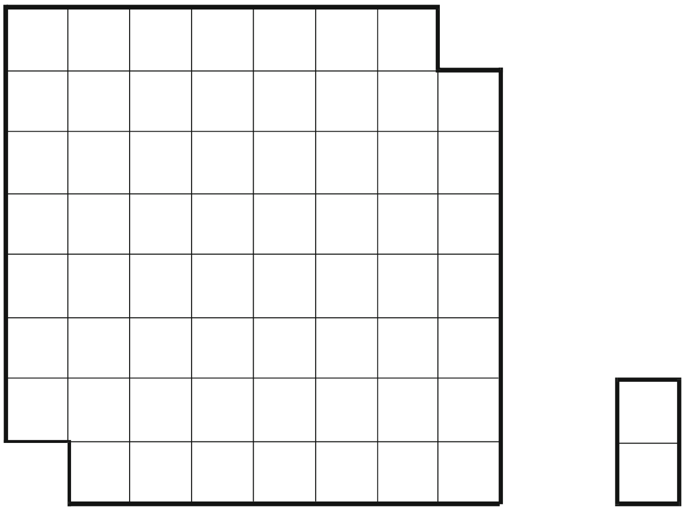


Fig. 1.7 .

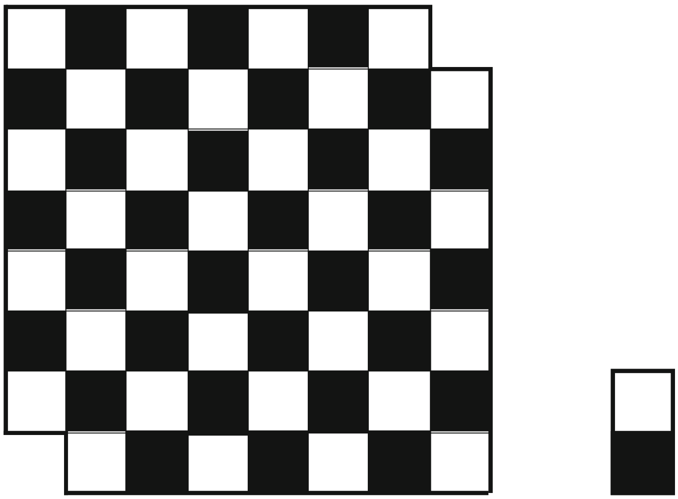
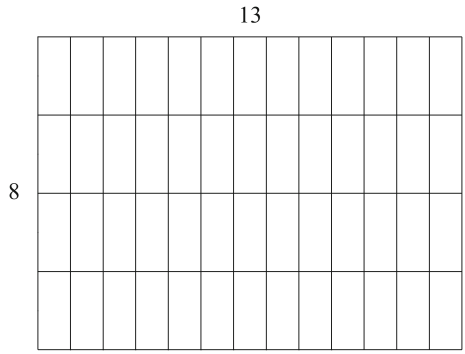


Fig. 1.8 .

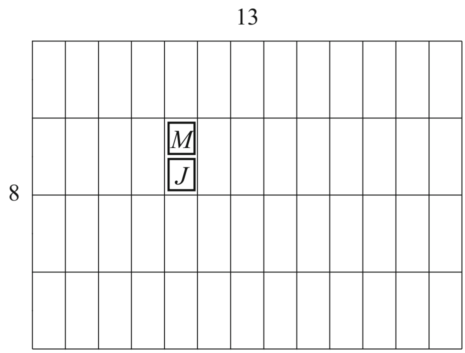
The Tiling Game (**Anti-method**, 6th Colorado Mathematical Olympiad, Soifer 1989). Mark and Julia are playing the following tiling game on a 1988 $\times$ 1989 chessboard. They in turn are putting 1 $\times$ 1 square tiles on the board. After each of them made exactly 100 moves (and thus they

Fig. 1.9 .



Tiling template *T* for a 8 x 13 chessboard

Fig. 1.10 Winning strategy



covered 200 squares of the board) a winner is determined as follows: Julia wins if the tiling of the board can be completed with dominoes. Otherwise Mark wins. (Dominoes are 1×2 rectangles, which cover exactly two squares of the board.) Can you find a strategy for one of the players allowing him to win regardless of what the moves of the other player may be? You cannot? Let me help you: Mark goes first!

Solution. Julia (i.e., the second player) has a strategy that allows her to win regardless of what Mark's moves may be. All she needs is a bit of home preparation: Julia creates a tiling template showing one particular way, call it *T*, of tiling the whole 1988×1989 chessboard with dominoes. Figure 1.9 shows one such tiling template *T* for an 8×13 chessboard.

The strategy for Julia is now clear. As soon as Mark puts a 1×1 tile *M* on the board, Julia puts her template *T* on the board to determine which domino of the template *T* contains Mark's tile *M*. She then puts her 1×1 tile *J* to cover the second square of the same domino (Fig. 1.10). When each player makes 100 moves, 100 dominoes will be covered, and the template T will show how to complete the tiling of the board. ■

5. Synthesis and Combinatorial Geometry

Secondary school mathematics consists predominantly of problems with single-idea solutions, found by analysis. We ought to introduce a sense of *mathematical reality* in the classroom by presenting *synthesis*, by offering problems that require for their solution ideas from a number of mathematical disciplines: geometry, algebra, number theory, trigonometry, linear algebra, etc.

And here comes *Combinatorial Geometry* to the rescue. It offers an abundance of problems that sound like a 'regular' secondary school geometry, but require for their solutions synthesis of ideas from geometry, algebra number theory, trigonometry, ideas of analysis, etc. See for example (Soifer 2009–1; 2009–3; 2011–1). Moreover, combinatorial geometry offers us open-ended problems. It offers problems that any geometry student can understand, and yet no one has yet solved! Let us stop this discrimination of our students based on their young age, and allow them to touch and smell, and work on 'real' mathematics and its unsolved problems. They may find a partial advance into solutions; they may settle some open problems completely. And they will then know the answer to what ought to become the fundamental questions of mathematical instruction: What is mathematics? What do mathematicians do?

In fact, I would opine that every discipline is about problem solving. And so the main goal of every discipline ought to be to enable students to learn *how to think within the discipline*, *how to solve problems of the discipline*, and finally *what that discipline is about*, and *what the professionals within the discipline do*. And mathematics to all sciences does what gymnastic does to all sports: Mathematics is gymnastics of the mind. Doing mathematics develops a universal approach to problem solving and intuition that go a long way in preparing our students for solving problems they will face in their lives.

6. Open Ended and Open Problems

As a junior at the university, I approached my supervisor Professor Leonid Yakovlevich Kulikov with an open problem I liked—he was my supervisor ever since my freshman year. He replied, "Learn first, the time will come later to enter research." He meant well, but politically speaking, this was a discrimination based on my young age. Seeing my disappointment, Kulikov continued, "It does not look like I can stop you from doing research. All right, whatever results you obtain on this open problem, I will count as your course paper." Soon I received my first research results, and my life in mathematics began.

We ought to allow our students to learn what mathematicians do by offering them not just unrelated to each other exercises but rather series of problems leading to a deeper and deeper understanding. And we ought to let students 'touch' unsolved problems of mathematics, give them a taste of the unknown, a taste of adventure and discovery. Combinatorial geometry serves these goals well by providing us with easy-to-understand, hard-to-solve—or even unsolved—problems. I will formulate here two examples. You can find their solutions in my Springer books listed in references.

Points in a Triangle (Soifer 2009–3). Out of any n points in or on the boundary of a triangle of area 1, there are 3 points that form a triangle of area at least ¼.

(a) Prove this statement for $n = 9$.

(b) Prove this statement for $n = 7$.

(c) Prove this statement for $n = 5$.

(d) Show that the statement is not true for $n = 4$, thus making $n = 5$ best possible.

Chromatic Number of the Plane (Soifer 2009–1). No matter how the plane is colored in n colors, there are two points of the same color distance 1 apart.

(a) Prove this statement for $n = 2$.

(b) Prove this statement for $n = 3$.

(c) Disprove this statement for $n = 7$.

(d) The answer for $n = 4$, 5, and 6 is unknown to man—this is a forefront of mathematics!

7. Beauty of 'Real' Mathematics Can Be Transplanted to Olympiads for Secondary Schools

New Olympiad problems occur to us in mysterious ways. This problem came to me one summer morning of 2003 as I was reading a never published

1980s manuscript of a Ramsey Theory monograph, while sitting by a mountain lake in Bavarian Alps. It all started with my finding a hole in a lemma, which prompted a construction of a counterexample (part b of the present problem). Problem (a) is a corrected particular case of that lemma, translated, of course, into a language of a nice 'real' story of a chess tournament. I found three distinct striking solutions of (a) and an even more special solution of (b). As a result, this problem became for me the most beautiful Olympiad problem I have ever created. What is more, the journey that led me from Ramset Theory to problems of mathematical Olympiads, continued to a mathematical problem of Finite Projective Planes! I will show here only one of my three solutions—read other solutions in (Soifer 2004) or (Soifer 2017).

Chess 7 × 7 (21st Colorado Mathematical Olympiad, April 16, 2004, A. Soifer).

(a) Each member of two 7-member chess teams is to play once against each member of the opposing team. Prove that as soon as 22 games have been played, we can choose 4 players and seat them at a round table so that each pair of neighbors has already played.

(b) Prove that 22 is the best possible; i.e., after 21 games the result of (a) cannot be guaranteed.

Solution. This solution harnesses the power of combinatorics. In the selection and editing process, Col. Dr. Bob Ewell suggested to use a 7 × 7 table to record the games played. We number the players in both teams. For each player of the first team we allocate a row of the table, and for each player of the second team a column. We place a checker on the table in the

Fig. 1.11 .